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To cite this article:

Devavrat Shah, Tauhid Zaman (2016) Finding Rumor Sources on Random Trees. *Operations Research* 64(3):736-755. <https://doi.org/10.1287/opre.2015.1455>

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Finding Rumor Sources on Random Trees

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We consider the problem of detecting the source of a rumor which has spread in a network using only observations about which set of nodes are infected with the rumor and with no information as to *when* these nodes became infected. In a recent work (Shah and Zaman 2010), this rumor source detection problem was introduced and studied. The authors proposed the graph score function *rumor centrality* as an estimator for detecting the source. They establish it to be the maximum likelihood estimator with respect to the popular Susceptible Infected (SI) model with exponential spreading times for regular trees. They showed that as the size of the infected graph increases, for a path graph (2-regular tree), the probability of source detection goes to 0 and for d -regular trees with $d \geq 3$ the probability of detection, say α_d , remains bounded away from 0 and is less than $1/2$. However, their results stop short of providing insights for the performance of the rumor centrality estimator in more general settings such as irregular trees or the SI model with nonexponential spreading times.

This paper overcomes this limitation and establishes the effectiveness of rumor centrality for source detection for generic random trees and the SI model with a generic spreading time distribution. The key result is an interesting connection between a continuous time branching process and the effectiveness of rumor centrality. Through this, it is possible to quantify the detection probability precisely. As a consequence, we recover all previous results as a special case and obtain a variety of novel results including the *universality* of rumor centrality in the context of tree-like graphs and the SI model with a generic spreading time distribution.

Keywords: rumors; networks; source detection; information diffusion.

Subject classifications: networks/graphs: theory; probability: applications.

Area of review: Special Issue on Information and Decisions in Social and Economic Networks.

History: Received June 2015; revisions received March 2015, September 2015; accepted October 2015. Published online in *Articles in Advance* February 23, 2016.

1. Introduction

Imagine someone starts a rumor which then spreads through a social network. After the rumor has spread for a long amount of time, we observe this network of rumor infected individuals. We only know who has heard the rumor and the underlying network structure. No information is given about *when* the people heard the rumor. Our goal is to use only this information to discover the source of the rumor.

This *rumor source detection problem* is very general and arises in many different contexts. For example, the rumor could be a computer virus spreading through the Internet, a contagious disease infecting a human population, or a trend or new product diffusing through a social network. In each of these different scenarios, detection of the source is of great interest. One would naturally like to find the originator of a malicious cyber-attack. Detecting the source of a viral epidemic would aid with the development of effective vaccination, quarantine, and prevention strategies. In social networks, sources of rumors, trends, or new product adoption may be effective at disseminating information, and their identification would be of interest to companies wishing to develop viral marketing campaigns.

Detection of the source is made challenging in each of these situations by the fact that one may not have information regarding the time of the infection or adoption. For example, if the computer virus remains dormant and then upon activation renders the system inoperable, it may not be possible to determine when the machine was infected. For contagious diseases, determining exactly when a person became infected can be difficult due to lack of sufficient data. Rather, only a broad time window of when the infection occurred may be known. For trends or new product adoption one may be able to determine the exact time of adoption if this occurs through a social network such as Facebook which records the time of each user's activity. However, there can be situations where people do not share the fact that they have adopted until much after they have done so, making it difficult to pinpoint precisely when the adoption occurred.

Given the wide ranging applications, it begs to understand the fundamental limitations of the source detection problem. Concretely, there are two key questions that need to be addressed. First, how does one actually construct the rumor source estimator? Since no information about infection times is given, a rumor source estimator would need to extract

all information about the identity of the source using only the structure of the rumor infected network, but it is not obvious in what manner. Second, what are the fundamental limits to this rumor source detection problem? In particular, how accurately can one find the rumor source, what is the magnitude of errors made in this detection, and how does the network structure affect one's ability to find the rumor source?

1.1. Related Work

Rumor spreading was originally studied in the context of epidemiology to predict, control, and prevent the spread of infectious diseases. The epidemiological models for the spread of disease generally consisted of individuals that could be in one of three states: susceptible, infected, or recovered. In the susceptible-infected-recovered or SIR model all three states are allowed, but there are variants such as the SI model which only consider susceptible and infected individuals. Daniel Bernoulli developed the first differential equation models for the spread of a disease (Bernoulli and Blower 2004). Modern differential equation models were introduced in (Kermack and McKendrick 1927) and later expanded in (Anderson and May 1979a) and (Anderson and May 1979b). These models provided insight into the disease spreading dynamics, but they were very coarse and made several simplifying assumptions about human populations. The next level of modeling involved taking into account the network over which the disease spread. Contact network modeling was able to capture in greater detail the specific manner by which disease spread. These models have allowed researchers to understand how the network structure affects the ability of a disease to become an epidemic (Moore and Newman 2000, Pastor-Satorras and Vespignani 2001, Ganesh et al. 2005). The insights obtained from modeling disease spreading at a network level have allowed epidemiologists to develop vaccination and quarantine strategies to control modern viral epidemics (Meyers et al. 2003, 2005; Pourbohouli et al. 2005, 2009; Bansal et al. 2006, 2010; Fraser et al. 2009; Yang et al. 2009).

The network models developed for disease propagation have found application in the context of online social networks. In Domingos and Richardson (2001), Kempe et al. (2003), and Hartline et al. (2008) optimization methods were applied to network models to select the best set of users to seed with a new product or information to maximize its spread in a social network. This work is complementary to that in epidemiology, where the goal is to prevent the spread of a viral outbreak, not accelerate it. Another interesting line of work has focused on using the spread of rumors in a social network to reconstruct the unknown network structure (Gomez-Rodriguez et al. 2010, Myers and Leskovec 2010, Netrapalli and Sanghavi 2012).

Controlling the spread of a rumor, whether it be a contagious disease or the adoption of a new product, has been the main focus of a large amount of research, but the question of identifying the source of the rumor has been

largely overlooked. A problem located at the intersection of probability theory and information theory recently emerged which is thematically related to rumor source detection. It is known as the *reconstruction problem* and the goal is to estimate the information possessed by a source based on noisy observations about this information as it propagates through a network. There are interesting similarities between the two problems: the signal of interest, the information of the source (for the reconstruction problem) and the rumor source itself (for the rumor source detection problem) are extremely “low-dimensional.” However, the observations for each problem, the noisy versions of the information (reconstruction) and infected nodes (rumor source detection), lie in a very “high-dimensional” setting. This makes estimation and detection quite challenging. It is not surprising that results for the reconstruction problem, even for tree or tree-like graphs, have required sophisticated mathematical techniques (Evans et al. 2000, Mossel 2001, Gerschenfeld and Montanari 2007). Therefore, one would expect similar types of challenges for the rumor source detection problem, which involves not estimating information at a known source, but rather finding the source itself among a large number of vertices in a network.

The rumor source detection problem was first formally posed and studied in Shah and Zaman (2010). The authors proposed a graph-score function called *rumor centrality* as an estimator for the rumor source. They showed that the node with maximal rumor centrality is the maximum likelihood (ML) estimate of the source for rumor spreading on regular trees under the SI model with homogeneous exponential spreading times. They demonstrated the effectiveness of this estimator by establishing that the rumor source is found with strictly positive probability for regular trees and geometric trees under this setting. The model and precise results from Shah and Zaman (2010) are described in §2.

Although this work laid the foundations of the rumor source detection problem, the results had some key limitations. First, they do not quantify the exact detection probability, say α_d , for d -regular graphs, for the proposed ML estimator other than $\alpha_2 = 0$, $\alpha_3 = 0.25$ and $0 < \alpha_d \leq 0.5$ for $d \geq 4$ for the SI model with exponential spreading times. Second, the results do not quantify the magnitude of the error in the event of not being able to identify the source. Third, the results do not provide any insights into how the estimator behaves for rumor spreading on generic heterogeneous tree (or tree-like) graphs under the SI model with a generic spreading time distribution.

1.2. Summary of Results

The primary reason behind the limitations of the results in Shah and Zaman (2010) is the fact that the analytic method employed there is quite specific to regular trees with homogeneous exponential spreading times. To overcome these limitations, as the main contribution of this work we introduce a novel analysis method that utilizes connections to the classical Markov branching process (MBP) (equivalently,

a generalized Polya's urn (GPU)). As a consequence of this, we are able to quantify the probability of the error event precisely and thus eliminate the shortcomings of the prior work.

Our results in this work collectively establish that, even though, rumor centrality is an ML estimator only for regular trees and the SI model with exponential spreading times, it is universally effective with respect to heterogeneity in the tree structure and spreading time distributions. Its effectiveness for generic random trees immediately implies its utility for finding sources in sparse random graphs that are locally tree-like. Examples include Erdős-Rényi and random regular graphs. A brief discussion to this effect can be found in §3.4.

The following is a summary of our main results (see §3 for precise statements):

1. *Regular trees, SI model with exponential spreading times:* We characterize α_d , the detection probability for d -regular trees, for all d . Specifically, for $d \geq 3$

$$\alpha_d = dI_{1/2}\left(\frac{1}{d-2}, \frac{d-1}{d-2}\right) - (d-1).$$

In above $I_x(a, b)$ is the incomplete beta function with parameters a, b evaluated at $x \in [0, 1]$ (see (4)). This implies that $\alpha_d > 0$ for $d \geq 3$, $\alpha_3 = 0.25$, and $\alpha_d \rightarrow 1 - \ln 2$ as $d \rightarrow \infty$. Furthermore, we show that the probability of rumor centrality estimating the k th infected node as the source decays as $\exp(-\Theta(k))$. The precise results are stated as Theorem 3.1, Corollaries 1 and 2.

2. *Generic random trees, SI model with exponential spreading times:* For generic random trees (see §3.2 for precise definition) which are expanding, we establish that there is strictly positive probability of correct detection using rumor centrality. Furthermore, the probability of rumor centrality estimating the k th infected node as the source decays as $\exp(-\Theta(k))$. The precise results are stated as Theorems 3.2 and 3.3.

3. *Geometric trees, SI model with generic spreading times:* For any geometric tree (see §3.2.2 for precise definition), we establish that the probability of correct detection goes to 1 as the number of infected nodes increases. The precise result is stated as Theorem 3.4.

4. *Generic random trees, SI model with generic spreading times:* For generic expanding random trees with generic spreading times (see §3.2 for definition), we establish that the probability of correct source detection remains bounded away from 0. The precise result is stated as Theorem 3.2.

2. Model, Problem Statement and Rumor Centrality

We start by describing the model and problem statement followed by a quick recall of the precise results from Shah and Zaman (2010). In the process, we shall recall the definition of rumor centrality and source estimation as introduced in Shah and Zaman (2010).

2.1. Model

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a possibly infinite connected graph. Let $v \in \mathcal{V}$ be a rumor source from which a rumor starts spreading at time 0. As per the classical Susceptible Infected (SI) model the rumor spreads in the graph. Specifically, each edge $e = (u_1, u_2)$ has a spreading time S_e associated with it. If node u_1 gets infected at time t_1 , then at time $t_1 + S_e$ the infection spreads from u_1 to u_2 . A node, once becoming infected, remains infected. The spreading times associated with edges are independent random variables with identical distribution. Let $F: \mathbb{R} \rightarrow [0, 1]$ denote the cumulative density function of the spreading time distribution. We shall assume that the distribution is non-negative valued, i.e., $F(0) = 0$ and it is nonatomic at 0, i.e., $F(0^+) = 0$. Since it is a cumulative density function, it is nondecreasing and $\lim_{x \rightarrow \infty} F(x) = 1$. The simplest, homogeneous SI model has exponential spreading times with parameter $\lambda > 0$ with $F(x) = 1 - \exp(-\lambda x)$ for $x \geq 0$. In Shah and Zaman (2010), the results were restricted to this homogeneous exponential spreading time setting. In this paper, we shall develop results for arbitrary spreading time distributions consistent with the above assumptions.

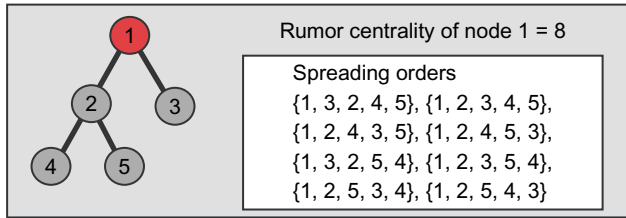
Given the above spreading model, we observe the rumor infected graph $G(t) = (V(t), E(t))$ at some time $t > 0$. To simplify our notation, we will refer to the time dependent rumor infected graph at time t simply as $G = (V, E)$. We do not know the value of t or the realization of the spreading times on edges $e \in E$; we only know the rumor infected nodes $V \subset \mathcal{V}$ and edges between them $E = V \times V \cap \mathcal{E}$. The goal is to find the rumor source (among V) given G .

We note here that in this setting we do not observe the underlying graph \mathcal{G} . This means we do not observe edges on the boundary between infected and noninfected nodes. However, these boundary edges do provide additional information. For example, if an infected node has a large number of uninfected neighbors, then it is likely that this node has not been infected for very long, otherwise more of its neighbors would be infected. Intuitively, this would mean that it is less likely that this node is the source. Our rumor source estimator, which we present next, does not require any knowledge of \mathcal{G} , though our analysis of the estimator's performance will require knowledge of the structure of \mathcal{G} . We will find that without observing \mathcal{G} , our rumor source estimator is still able to perform well on a variety of graphs under general spreading models.

2.2. Rumor Centrality: An Estimator

To solve the rumor source detection problem, the notion of rumor centrality was introduced in Shah and Zaman (2010). Rumor centrality is a "graph score" function. That is, it takes $G = (V, E)$ as input and assigns a non-negative number or score to each of the vertices. Then the estimated source is the one with maximal (ties broken uniformly at random) score or rumor centrality. The node with maximal rumor centrality is called the "rumor center" (which is also the

Figure 1. (Color online) Example of rumor centrality calculation for a 5 node network.



Note. The rumor centrality of node 1 is 8 because there are 8 spreading orders that it can originate, which are shown in the figure.

estimated source) with ties broken uniformly at random. We start with the precise description of rumor centrality for a tree¹ graph G : the rumor centrality of node $u \in V$ with respect to $G = (V, E)$ is

$$R(u, G) = \frac{|V|!}{\prod_{w \in V} T_w^u}, \quad (1)$$

where T_w^u is the size of the subtree of G that is rooted at w and points away from u . For example, in Figure 1, let u be node 1. Then $|V| = 5$; the subtree sizes are $T_1^1 = 5$, $T_2^1 = 3$, $T_3^1 = T_4^1 = T_5^1 = 1$ and hence $R(1, G) = 8$. In Shah and Zaman (2010), a linear time algorithm is described to compute the rumor centrality of all nodes building on the relation $R(u, G)/R(v, G) = T_u^v/T_v^u$ for neighboring nodes $u, v \in V ((u, v) \in E)$.

The rumor centrality of a given node $u \in V$ for a tree given by (1) is precisely the number of distinct spreading orders that could lead to the rumor infected graph G starting from u . This is equivalent to computing the number of linear extensions of the partial order imposed by the graph G due to causality constraints of rumor spreading. Under the SI model with homogeneous exponential spreading times and a regular tree, it turns out that each of the spreading orders is equally likely. Therefore, rumor centrality turns out to be the maximum likelihood (ML) estimator for the source in this specific setting (cf. Shah and Zaman 2010). In general, the likelihood of each node $u \in V$ being the source given G is proportional to the weighted summation of the number of distinct spreading orders starting from u , where weight of a spreading order could depend on the details of the graph structure and spreading time distribution of the SI model. Now for a tree graph and SI model with homogeneous exponential spreading times, as mentioned above, such a quantity can be computed in linear time. But in general, this could be complicated. For example, computing the number of linear extensions of a given partial order is known to be #P-complete (Brightwell and Winkler 1991). There are algorithms for approximately sampling linear extensions given a partial order (Karzanov and Khachiyan 1991), Shah and Zaman (2010) proposed the following simpler alternative for general graphs.

DEFINITION 1 (RUMOR CENTRALITY). Given node $u \in V$ in graph $G = (V, E)$, let $T \subset G$ denote a breadth-first search tree of u with respect to G . Then, the rumor centrality of u with respect to G is obtained by computing it as per (1) with respect to T . The estimated rumor source is the one with maximal rumor centrality (ties broken uniformly at random).

2.3. Prior Results

In Shah and Zaman (2010), the authors established that rumor centrality is the maximum-likelihood estimator for the rumor source when the underlying graph \mathcal{G} is a regular tree. They studied the effectiveness of this ML estimator for such regular trees. Specifically, suppose we observe the $n(t)$ node rumor infected graph G after time t , which is a subgraph of \mathcal{G} . Let C_t^k be the event that the source estimated as per rumor centrality is the k th infected node, and thus C_t^1 corresponds to the event of correct detection. The following are key results from Shah and Zaman (2010):

THEOREM 2.1 (SHAH AND ZAMAN 2010). Let \mathcal{G} be a d -regular infinite tree with $d \geq 2$. Let

$$\alpha_d^L = \liminf_{t \rightarrow \infty} \mathbf{P}(C_t^1) \leq \limsup_{t \rightarrow \infty} \mathbf{P}(C_t^1) = \alpha_d^U. \quad (2)$$

Then,

$$\alpha_2^L = \alpha_2^U = 0, \quad \alpha_3^L = \alpha_3^U = \frac{1}{4}, \quad \text{and} \quad (3)$$

$$0 < \alpha_d^L \leq \alpha_d^U \leq \frac{1}{2}, \quad \forall d \geq 4.$$

3. Main Results

We state the main results of this paper. In a nutshell, our results concern the characterization of the probability of C_t^k for any $k \geq 1$ for large t when \mathcal{G} is a generic tree. As a consequence, it provides a characterization of the performance for sparse random graphs.

3.1. Regular Trees, SI Model with Exponential Spreading Times

We first look at rumor source detection on regular trees with degree $d \geq 3$, where rumor centrality is an exact ML estimator when the spreading times are exponentially distributed. Our results will utilize properties of Beta random variables. We recall that the regularized incomplete Beta function $I_x(a, b)$ is the probability that a Beta random variable with parameters a and b is less than $x \in [0, 1]$,

$$I_x(a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^x t^{a-1} (1-t)^{b-1} dt, \quad (4)$$

where $\Gamma(\cdot)$ is the standard Gamma function. For regular trees of degree ≥ 3 we obtain the following result.

THEOREM 3.1. Let \mathcal{G} be d -regular infinite tree with $d \geq 3$. Assume a rumor spreads on \mathcal{G} as per the SI model with exponential distribution with rate λ . Then, for any $k \geq 1$,

$$\lim_{t \rightarrow \infty} \mathbf{P}(C_t^k) = I_{1/2} \left(k - 1 + \frac{1}{d-2}, 1 + \frac{1}{d-2} \right) + (d-1) \left(I_{1/2} \left(\frac{1}{d-2}, k + \frac{1}{d-2} \right) - 1 \right). \quad (5)$$

For $k = 1$, Theorem 3.1 yields that $\alpha_d^L = \alpha_d^U = \alpha_d$ for all $d \geq 3$ where

$$\alpha_d = dI_{1/2}\left(\frac{1}{d-2}, \frac{d-1}{d-2}\right) - (d-1). \tag{6}$$

More interestingly,

COROLLARY 1.

$$\lim_{d \rightarrow \infty} \alpha_d = 1 - \ln 2 \approx 0.307. \tag{7}$$

For any $d \geq 3$, we can obtain a simple upper bound for Theorem 3.1 which provides the insight that the probability of error in the estimation decays exponentially with error distance (not number of hops in graph, but based on chronological order of infection) from the true source.

COROLLARY 2. When \mathcal{G} is a d -regular infinite tree, for any $k \geq 1$,

$$\lim_{t \rightarrow \infty} \mathbf{P}(C_t^k) \leq k(k+1) \left(\frac{1}{2}\right)^{k-1} \asymp \exp(-\Theta(k)).$$

To provide intuition, we plot the asymptotic error distribution $\lim_{t \rightarrow \infty} \mathbf{P}(C_t^k)$ for different degree regular trees in Figure 2. As can be seen, for degrees greater than 4, all the error distributions fall on top of each other, and the probability of detecting the k th infection as the source decays exponentially in k . We also plot the upper bound from Corollary 2. As can be seen, this upper bound captures the rate of decay of the error probability. Thus we see tight concentration of the error for this class of graphs. Figure 3 plots the asymptotic correct detection probability α_d versus degree d for these regular trees. It can be seen that the detection probability starts at $1/4$ for degree 3 and rapidly converges to $1 - \ln(2)$ as the degree goes to infinity.

Figure 2. (Color online) $\lim_{t \rightarrow \infty} \mathbf{P}(C_t^k)$ vs. k for regular trees of different degree.

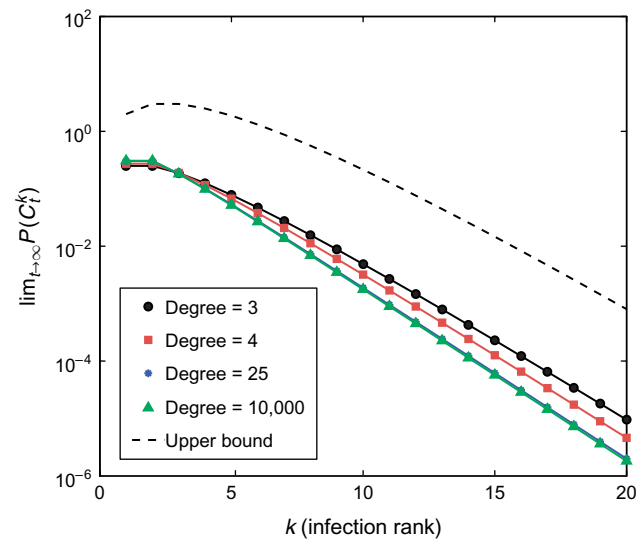
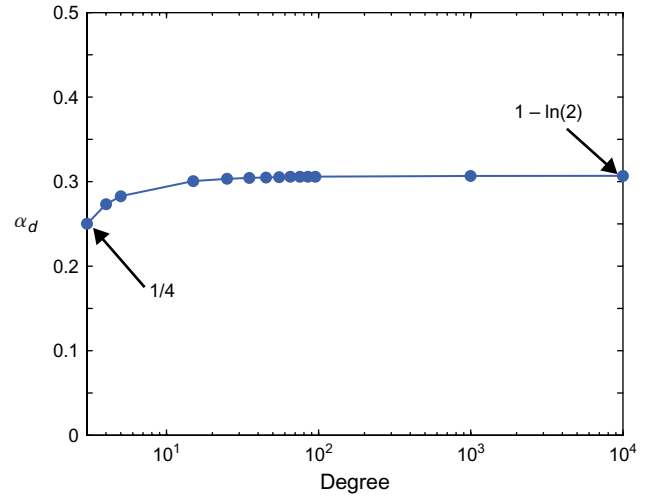


Figure 3. (Color online) α_d vs. degree d for regular trees.



3.2. Generic Random Trees, SI Model with Generic Spreading Times

The above precise results were obtained using the memoryless property of the exponential distribution and the regularity of the trees. Next, we wish to look at a more general setting both in terms of tree structures and spreading time distributions. In this more general setting, although we cannot obtain precise values for the detection and error probabilities, we are able to make statements about the nontriviality of the detection probability of rumor centrality. When restricted to exponential spreading times for generic trees, we can identify bounds on the error probability as well. Let us start by defining what we mean by generic random trees through a generative model.

DEFINITION 2 (GENERIC RANDOM TREES). It is a rooted random tree, generated as follows: given a root node as a starting vertex, add η_0 children to root where η_0 is an independent random variable with distribution \mathcal{D}_0 . If $\eta_0 \neq 0$, then add a random number of children chosen as per distribution \mathcal{D} over $\{0, 1, \dots\}$ independently to each child of the root. Recursively, to each newly added node, add independently a random number of nodes as per distribution \mathcal{D} .

The generative model described above is precisely the standard Galton-Watson branching process if $\mathcal{D}_0 = \mathcal{D}$. If we take \mathcal{D}_0 and \mathcal{D} to be deterministic distributions with support on d and $d - 1$ respectively, then it gives the d -regular tree. For a random d -regular graph on n nodes, as n grows the neighborhood of a randomly chosen node in the graph converges (in distribution, locally) to such a d -regular tree. If we take $\mathcal{D}_0 = \mathcal{D}$ as a Poisson distribution with mean $c > 0$, then it asymptotically equals (in distribution) to the local neighborhood of a randomly chosen node in a sparse Erdős-Rényi graph as the number of nodes grows. Recall that a (sparse) Erdős-Rényi graph on n nodes with parameter c is generated by selecting each of the $\binom{n}{2}$ edges to be present

with probability c/n independently. Effectively, random trees as described above capture the local structure for sparse random graphs reasonably well. For that reason, establishing the effectiveness of rumor centrality for source detection for such trees provide insights into its effectiveness for sparse random graph models.

We shall consider spreading time distributions to be generic. Let $F: [0, \infty) \rightarrow [0, 1]$ be the cumulative distribution function of the spreading times. Clearly $F(0) = 0$, F is nondecreasing and $\lim_{t \rightarrow \infty} F(t) = 1$. In addition, we shall require that the distribution is nonatomic at 0, i.e., $F(0^+) = 0$. We state the following result about the effectiveness of rumor centrality with such generic spreading time distribution.

THEOREM 3.2. *Let η_0 , distributed as \mathcal{D}_0 , be such that $\Pr(\eta_0 \geq 3) > 0$ and let η , distributed as per \mathcal{D} , be such that $1 < \mathbf{E}[\eta] < \infty$. Suppose the rumor starts from the root of the random tree generated as per distributions \mathcal{D}_0 and \mathcal{D} as described above and spreads as per the SI model with a spreading time distribution with an absolutely continuous density. Then,*

$$\liminf_{t \rightarrow \infty} \mathbf{P}(C_t^1) > 0.$$

The above result says that irrespective of the structure of the random trees, spreading time distribution and elapsed time, there is nontrivial probability of detecting the root as the source by rumor centrality. The interesting aspect of the result is that this nontrivial detection probability is established by studying events when the tree grows without bound. For finite size trees with n nodes, the rumor source can be estimated by selecting a random node, giving a probability of correct detection of $n^{-1} > 0$. However, such events are trivial and are not of much interest to us (neither mathematically, nor motivationally).

3.2.1. Generic Random Trees, SI Model with Exponential Spreading Times. Extending the results of Theorem 3.2 for explicitly bounding the probability of the error event $\mathbf{P}(C_t^k)$ for generic spreading time distribution seems rather challenging. Here we provide a result for generic random trees with exponential spreading times.

THEOREM 3.3. *Consider the setup of Theorem 3.2 with spreading times being homogeneous exponential distributions with (unknown, but fixed) parameter $\lambda > 0$. In addition, let $\mathcal{D}_0 = \mathcal{D}$. Let η , distributed as per \mathcal{D} , be such that $\mathbf{E}[\eta] > 1$ and $\mathbf{E}[\exp(\theta\eta)] < \infty$ for all $\theta \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. Then, for appropriate constants $C', C'' > 0$,*

$$\limsup_{t \rightarrow \infty} \mathbf{P}(C_t^k) \leq C' \exp(-kC''). \quad (8)$$

The above result establishes an explicit upper bound on the probability of the error event. The bound applies to essentially any generic random tree and demonstrates that the probability of identifying later infected nodes as the rumor source decreases exponentially fast.

3.2.2. Geometric Trees, SI Model with Generic Spreading Times. The trees considered thus far, d -regular trees with $d \geq 3$ or random trees with $\mathbf{E}[\eta] > 1$, grow exponentially in size with the diameter of the tree. This is in contrast with path graphs or d -regular trees with $d = 2$ which grow only linearly in diameter. It can be easily seen that the probability of correct detection, $\mathbf{P}(C_t^1)$ will scale as $\Theta(1/\sqrt{t})$ for path graphs as long as the spreading time distribution has nontrivial variance (see Shah and Zaman 2010 for proof of this statement for the SI model with exponential spreading times). In contrast, the results of this paper stated thus far suggest that the expanding trees allow for nontrivial detection as $t \rightarrow \infty$. Thus, qualitatively path graphs and expanding trees are quite different—one does not allow detection whereas the other does. To understand where the precise detectability threshold lies, we look at polynomially growing geometric trees.

DEFINITION 3 (GEOMETRIC TREE). A geometric tree is a rooted, nonregular tree parameterized by constants α, b , and c , with $\alpha \geq 0$, $0 < b \leq c$, and root node v^* . Let d^* be the degree of v^* , let the neighbors of v^* be denoted v_1, v_2, \dots, v_{d^*} , and let the subtree rooted at v_i and directed away from v^* be denoted by T_i for $i = 1, 2, \dots, d^*$. Denote the number of nodes in T_i at distance exactly r from the subtree's root node v_i as $n^i(r)$. Then we require that for all $1 \leq i \leq d^*$

$$br^\alpha \leq n^i(r) \leq cr^\alpha. \quad (9)$$

The condition imposed by (9) states that each of the neighboring subtrees of the root should satisfy polynomial growth (with exponent $\alpha > 0$) and regularity properties. The parameter $\alpha > 0$ characterizes the growth of the subtrees and the ratio c/b describes the regularity of the subtrees. If $c/b \approx 1$ then the subtrees are somewhat regular, whereas if the ratio is much greater than 1, there is substantial heterogeneity in the subtrees. Note that the path graph is a geometric tree with $\alpha = 0$, $b = 1$, and $c = 2$.

We shall consider the scenario where the rumor starts from the root node of a rooted geometric tree. We shall show that rumor centrality detects the root as the source with an asymptotic probability of 1 for a generic spreading time distribution with exponential tails. This is quite interesting given the fact that rumor centrality is an ML estimator only for regular trees with exponential spreading times. The precise result is stated next.

THEOREM 3.4. *Let \mathcal{G} be a rooted geometric tree as described above with parameters $\alpha > 0$, $0 < b \leq c$ and root node v^* with degree d^* such that*

$$d_{v^*} > \frac{c}{b} + 1.$$

Suppose the rumor starts spreading on \mathcal{G} starting from v^ as per the SI model with a generic spreading time distribution whose cumulative density function $F: \mathbb{R} \rightarrow [0, 1]$ is such*

that (a) $F(0) = 0$, (b) $F(0^+) = 0$, and (c) if X is a random variable distributed as per F then $\mathbf{E}[\exp(\theta X)] < \infty$ for $\theta \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. Then

$$\lim_t \mathbf{P}(C_t^1) = 1.$$

A similar theorem was proven in Shah and Zaman (2010), but only for the SI model with exponential spreading times. We have now extended this result to arbitrarily distributed spreading times. Theorem 3.4 says that $\alpha = 0$ and $\alpha > 0$ serve as a threshold for nontrivial detection: for $\alpha = 0$, the graph is a path graph, so we would expect the detection probability to go to 0 as $t \rightarrow \infty$ as discussed above, but for $\alpha > 0$ the detection probability converges to 1 as $t \rightarrow \infty$.

3.3. Detection Probability and Graph Growth: Discussion

Our results can be viewed as relating detection probability to graph growth parametrized by α . For path graphs, where no detection is possible, $\alpha = 0$. For any finite, positive α we have geometric graphs where the detection probability converges to one. For regular trees or random graphs, the growth is exponential, which gives $\alpha = \infty$, and we have a detection probability that is strictly between zero and one.

To understand these results at a high level, it is helpful to consider the properties of the rumor center given by Lemma 1. Essentially, this lemma states that the graph is *balanced* around the rumor center. For the rumor source to be the rumor center (and therefore correctly identified as the true source), the rumor must spread in a balanced way. For a path graph ($\alpha = 0$), balance is a very delicate condition, requiring both subtrees of the source to be exactly equal in size. The probability of this occurring goes to zero as the graph size goes to infinity.

For any non-negative, finite alpha, this balance condition becomes easier to achieve if the source has degree greater than or equal to three. In this case, because the number of vertices grows polynomially, the variation of the size of a rumor infected subtree after a time t is much smaller than the expected value of its size, resulting in a concentration of the size. This means that with high probability, no subtree will be larger than half of the network size, and balance is achieved. The key here is that the boundary where the rumor can spread grows slower than the size of the rumor infected graph. If the graph has $d^{\alpha+1}$ nodes, then the boundary contains d^α nodes.

For infinite alpha, which corresponds to graphs with exponential growth, the rumor boundary size is of the same order of magnitude in size as the rumor infected graph. This results in a high variance in the subtree size. We would expect this high variance to result in detection becoming impossible. However, our analysis shows that the manner in which the rumor spreads on these graphs results in detection being possible with strictly positive probability. Another way to view this result is that the vertices in each subtree act

as witnesses which we can use to triangulate the source. If there are three or more subtrees, and the subtree sizes do not vary considerably (as in graphs with polynomial growth), then the witnesses have low noise, and we can detect the source exactly as the observed rumor infected graph grows. For exponentially growing graphs, the noise in the signals provided by the witnesses grows with the number of witnesses. The increased number of witnesses balances the increased noise to give a detection probability that remains strictly positive as the graph size goes to infinity.

3.4. Locally Tree-Like Graphs: Discussion

The results of the paper are primarily for tree structured graphs. On one hand, these are specialized graphs. On the other hand, they serve as local approximations for a variety of sparse random graph models. As discussed earlier, for a random d -regular graph over m nodes, a randomly chosen node's local neighborhood (say up to distance $o(\log m)$) is a tree with high probability. Similarly, consider an Erdős-Rényi graph over m nodes with each edge being present with probability $p = c/m$ independently for any $c > 0$ ($c > 1$ is an interesting regime due to the existence of a giant component). A randomly chosen node's local neighborhood (up to distance $o(\log m)$) is a tree and distributionally equivalent (in the large m limit) to a random tree with Poisson degree distribution.

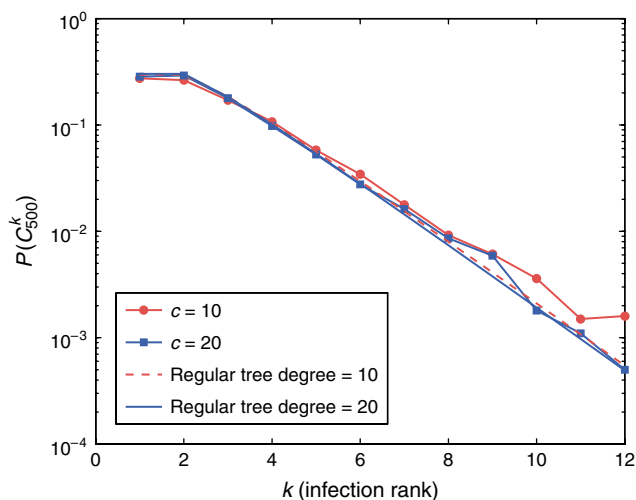
Given such "locally tree-like" structural properties, if a rumor spreads on a random d -regular graph or sparse Erdős-Rényi graph for time $o(\log m)$ starting from a random node, then rumor centrality can detect the source with guarantees given by Theorems 3.1 and 3.2. Thus, although the results of this paper are for tree structured graphs, they do have meaningful implications for tree-like sparse graphs.

For the purpose of illustration, we conducted some simulations for Erdős-Rényi graphs that are reported in Figure 4. We generated graphs with $m = 50,000$ nodes and edge probabilities $p = c/m$ for $c = 10$ and $c = 20$. The rumor graph contained $n = 500$ nodes and the spreading times had an exponential distribution with mean one. We used the general graph version of rumor centrality as defined in Definition 1 as the rumor source estimator. We ran 10,000 rumor spreading simulations to obtain the empirical error distributions plotted in Figure 4. As can be seen, the error drops of exponentially in k , very similar to the regular tree error distribution. To make this more evident, we also plot the asymptotic error distributions for regular trees of degree 10 and 20 and it can be seen that the error decays at similar, exponential rates. This indicates that even though there is substantial randomness in the graph, the asymptotic rumor source detection error distribution behaves as though it were a regular tree graph. This result also suggests that the bounds in Theorem 3.3 are loose for this graph.

4. Proofs

Here proofs of the results stated in §3 are presented. We establish results for d -regular trees by connecting rumor

Figure 4. (Color online) Empirical error probability vs. k for Erdős-Rényi graphs with 500 vertices, mean degree 10 and 20, and exponentially distributed spreading times with mean one.



Note. Also shown are $\lim_{T \rightarrow \infty} \mathbf{P}(C_i^k)$ for degree 10 and 20 regular trees with exponentially distributed spreading times with mean one.

spreading with Polya urn models and branching processes. Later we extend this novel method to establish results for generic random trees under arbitrary spreading time distributions. After this, we prove Theorem 3.4 using standard Chernoff’s bound and the polynomial growth property of geometric trees.

4.1. Proof of Theorem 3.1: d -Regular Trees

4.1.1. Background: Polya’s Urn. We will recall Polya’s urn process and its asymptotic properties that we shall crucially utilize in establishing Theorem 3.1. An interested reader can find a good exposition in Athreya and Ney (1972).

In the simplest form, Polya’s Urn process operates in discrete time. Initially, at time 0, an urn contains balls of two types, say W_0 white balls and B_0 black balls. Let W_n and B_n denote the number of white and black balls, respectively, at the end of time $n \geq 1$. At each time $n \geq 1$, a ball is drawn at random from the urn ($W_{n-1} + B_{n-1}$ balls in total). This ball is added back along with $\alpha \geq 1$ new balls of the same type leading to a new configuration of balls (W_n, B_n). For instance, at time n , a white ball is drawn with probability $W_{n-1}/(W_{n-1} + B_{n-1})$ and we have that $W_n = W_{n-1} + \alpha$, $B_n = B_{n-1}$.

Under the above described process, it is easy to check that the fraction of white (or black) balls is a bounded martingale. Therefore, by the martingale convergence theorem, it has a limit almost surely. What is interesting is that the limiting distribution is nicely characterized as stated below.

THEOREM 4.1 (ATHREYA AND NEY 1972, THEOREM 1, P. 220). For the Polya’s Urn process described above

$$\frac{W_n}{W_n + B_n} \rightarrow Y \quad \text{almost surely,} \quad (10)$$

where Y is a Beta random variable with parameters W_0/α and B_0/α . That is, for $x \in [0, 1]$,

$$P(Y \leq x) = I_x\left(\frac{W_0}{\alpha}, \frac{B_0}{\alpha}\right),$$

where $I_x(a, b)$ is the incomplete Beta function defined as in (4)

4.1.2. Setup and Notation. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an infinite d -regular tree and let the rumor start spreading from a node, say v_1 . Without loss of generality, we view the tree as a randomly generated tree, as described in §3, with v_1 being the root with d children and all the subsequent nodes with $d - 1$ children (hence each node has degree d). We shall be interested in $d \geq 3$. Now suppose the rumor is spread on this tree starting from v_1 as per the SI model with exponential distribution with rate $\lambda > 0$.

Initially, node v_1 is the only rumor infected node and its d neighbors are potential nodes that can receive the rumor. We will denote the set of nodes that are not yet rumor infected but are neighbors of rumor infected nodes as the *rumor boundary*. Initially the rumor boundary consists of the d neighbors of v_1 . Under the SI model, each edge has an independent exponential clock of mean $1/\lambda$. The minimum of d independent exponentials of mean $1/\lambda$ is an exponential random variable of mean $1/(d\lambda)$, and hence when one of the d nodes (chosen uniformly at random) in the rumor boundary gets infected, the infection time has an exponential distribution with mean $1/(d\lambda)$. Upon this infection, this node gets removed from the boundary and adds its $d - 1$ children to the rumor boundary. That is, each infection adds $d - 2$ new nodes to the rumor boundary. In summary, let $Z(t)$ denote the number of nodes in the rumor boundary at time t , then $Z(0) = d$ and $Z(t)$ evolves as follows: each of the $Z(t)$ nodes has an exponential clock of mean $1/\lambda$; when it ticks, it dies and $d - 1$ new nodes are born which in turn start their own independent exponential clocks of mean $1/\lambda$ and so on. Let u_1, \dots, u_d be the children of v_1 ; let $Z_i(t)$ denote the number of nodes in the rumor boundary that belong to the subtree $\mathcal{T}_i(t)$ that is rooted at u_i with $Z_i(0) = 1$ for $1 \leq i \leq d$; $Z(t) = \sum_{i=1}^d Z_i(t)$. Let $T_i(t) = |\mathcal{T}_i(t)|$ denote the total number of nodes infected in the subtree rooted at u_i at time t ; initially $T_i(0) = 0$ for $1 \leq i \leq d$. Since each infected node add $d - 2$ nodes to the rumor boundary, it can be easily checked that $Z_i(t) = (d - 2)T_i(t) + 1$ and hence $Z(t) = (d - 2)T(t) + d$ with $T(t)$ being the total number of infected nodes at time t (excluding v_1).

4.1.3. Probability of Correct Detection. Suppose we observe the rumor infected nodes at some time t which we do not know. That is, we observe the rumor infected graph $G(t)$ which contains the root v_1 and its d infected subtrees $\mathcal{T}_i(t)$ for $1 \leq i \leq d$. We recall the following result of Shah and Zaman (2010) that characterizes the rumor center (for a proof see §4.2).

LEMMA 1 (SHAH AND ZAMAN 2010). *Given a tree graph $G = (V, E)$, there can be at most two rumor centers. Specifically, a node $v \in V$ is a rumor center if and only if*

$$T_i^v \leq \frac{1}{2} \left(1 + \sum_{j \in \mathcal{N}(v)} T_j^v \right), \quad \forall i \in \mathcal{N}(v), \quad (11)$$

where $\mathcal{N}(v) = \{u \in V : (u, v) \in E\}$ are neighbors of v in G and T_j^v denotes the size of the sub-tree of G that is rooted at node $j \in \mathcal{N}(v)$ and includes all nodes that are away from node v (i.e., the subtree does not include v). The rumor center is unique if the inequality in (11) is strict for all $i \in \mathcal{N}(v)$.

This immediately suggests the characterization of the event that node v_1 , the true source, is identified by rumor centrality at time t : v_1 is a rumor center only if $2T_i(t) \leq 1 + \sum_{j=1}^d T_j(t)$ for all $1 \leq i \leq d$, and if the inequality is strict then it is the unique rumor center. Let $E_i = \{2T_i(t) < 1 + \sum_{j=1}^d T_j(t)\}$ and $F_i = \{2T_i(t) \leq 1 + \sum_{j=1}^d T_j(t)\}$. Then,

$$\begin{aligned} \mathbf{P}(C_i^1) &\geq \mathbf{P}\left(\bigcap_{i=1}^d E_i\right) = 1 - \mathbf{P}\left(\bigcup_{i=1}^d E_i^c\right) \\ &\stackrel{(a)}{\geq} 1 - \sum_{i=1}^d \mathbf{P}(E_i^c) \stackrel{(b)}{=} 1 - d\mathbf{P}(E_1^c). \end{aligned} \quad (12)$$

Above, (a) follows from the union bound of events and (b) from symmetry. Similarly, we have

$$\begin{aligned} \mathbf{P}(C_i^1) &\leq \mathbf{P}\left(\bigcap_{i=1}^d F_i\right) = 1 - \mathbf{P}\left(\bigcup_{i=1}^d F_i^c\right) \\ &\stackrel{(a)}{=} 1 - \sum_{i=1}^d \mathbf{P}(F_i^c) \stackrel{(b)}{=} 1 - d\mathbf{P}(F_1^c). \end{aligned} \quad (13)$$

Above, (a) follows because events F_1^c, \dots, F_d^c are disjoint and (b) from symmetry. Therefore, the probability of correct detection boils down to evaluating $\mathbf{P}(E_1^c)$ and $\mathbf{P}(F_1^c)$ which, as we shall see, will coincide with each other as $t \rightarrow \infty$. Therefore, the bounds of (12) and (13) will provide the exact evaluation of the correct detection probability as $t \rightarrow \infty$.

4.1.4. $\mathbf{P}(E_1^c)$, $\mathbf{P}(F_1^c)$ and Polya’s Urn. Effectively, the interest is in the ratio $T_1(t)/(1 + \sum_{i=1}^d T_i(t))$, especially as $t \rightarrow \infty$ (implicitly we are assuming that this ratio is well defined for a given t or else by definition there is only one node infected which will be v_1 , the true source). It can be easily verified that as $t \rightarrow \infty$, $T_i(t) \rightarrow \infty$ for all i almost surely and hence $Z_i(t) = (d - 2)T_i(t) + 1$ goes to ∞ as well. Therefore, it is sufficient to study the ratio $Z_1(t)/(\sum_{j=1}^d Z_j(t))$ as $t \rightarrow \infty$ since we shall find that this ratio converges to a random variable with density on $[0, 1]$. In summary, if we establish that the ratio $Z_1(t)/(\sum_{j=1}^d Z_j(t))$ converges in distribution on $[0, 1]$ with a well defined density, then it immediately follows that $\mathbf{P}(E_1^c) \xrightarrow{t \rightarrow \infty} \mathbf{P}(F_1^c)$ and we can use $Z_1(t)/(\sum_{j=1}^d Z_j(t))$ in place of $T_1(t)/(1 + \sum_{j=1}^d T_j(t))$.

With these facts in mind, let us study the ratio $Z_1(t)/(\sum_{j=1}^d Z_j(t))$. For this, it is instructive to view the simultaneous evolution of $(Z_1(t), Z_{\neq 1}(t))$ (where $Z_{\neq 1}(t) \triangleq \sum_{j=2}^d Z_j(t)$) as that induced by the standard, discrete time, Polya’s urn. Initially, $\tau_0 = 0$ and there is one ball of type 1 (white) representing $Z_1(\tau_0) = 1$ and $d - 1$ balls of type 2 (black) representing $Z_{\neq 1}(\tau_0) = d - 1$ in a given urn. The j th event happens at time τ_j (also known as a split time) when one of the $Z_1(\tau_{j-1}) + Z_{\neq 1}(\tau_{j-1}) (= d + (j - 1)(d - 2))$ balls chosen uniformly at random is returned to the urn along with $d - 2$ new balls of its type. If we set $\tau_j - \tau_{j-1}$ equal to an exponential random variable with mean $1/(\lambda(d + (j - 1)(d - 2)))$, then it is easy to check that the fraction of balls of type 1 is identical in law to that of $Z_1(t)/(\sum_{i=1}^d Z_i(t))$ (here we are using the *memoryless* property of exponential random variables crucially). Therefore, for our purposes, it is sufficient to study the limit law of fraction of balls of type 1 (or white) under this Polya’s urn model.

From the discussion in §4.1.1, it follows that the ratio $Z_1(t)/(\sum_{i=1}^d Z_i(t))$ converges to a Beta random variable with parameters $1/(d - 2)$ and $(d - 1)/(d - 2)$. Since the Beta distribution has a density on $[0, 1]$, from the above discussion it follows that as $t \rightarrow \infty$, $|\mathbf{P}(E_1^c) - \mathbf{P}(F_1^c)| \rightarrow 0$ and hence from (12), (13)

$$\lim_{t \rightarrow \infty} \mathbf{P}(C_i^1) = 1 - d \left(1 - I_{1/2} \left(\frac{1}{d - 2}, 1 + \frac{1}{d - 2} \right) \right), \quad (14)$$

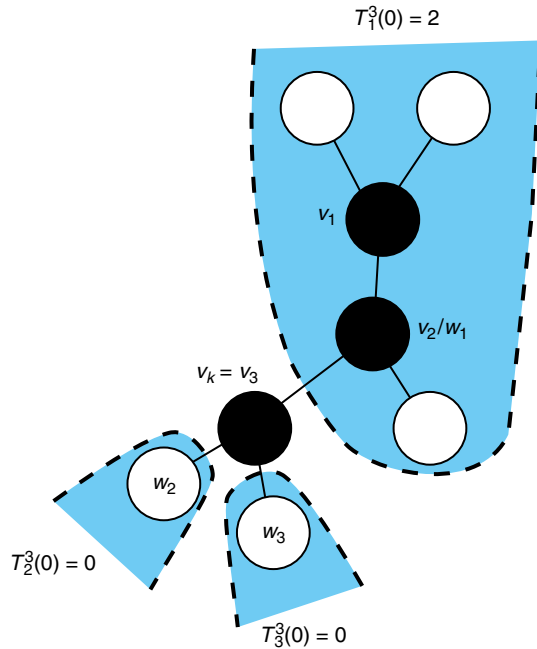
where $I_{1/2}(a, b)$ is the probability that a Beta random variable with parameters a and b takes value in $[0, 1/2]$. Note that this establishes the result of Theorem 3.1 for $k = 1$ in (5).

4.1.5. Probability of C_i^k . Thus far we have established Theorem 3.1 for $k = 1$ (the probability of the rumor center being the true source). The probability of the event C_i^k (the k th infected node being the rumor center) is evaluated in an almost identical manner with a minor difference. For this reason, we present an abridged version of the proof.

Let $T_k = \inf\{t : n(t) = k\}$ represent the time when k th node is infected. It can be easily checked that for $d \geq 3$ regular tree with exponential spreading time distribution, $T_k < \infty$ with probability 1. Consider $t \geq T_k$. For $k \geq 2$, let v_k be the k th infected node when the rumor starts from v_1 . We will evaluate the probability of identifying v_k as the rumor center. Let G represent the rumor infected tree observed at time t with $n(t) \geq k$ nodes. Let w_1, \dots, w_d be the d neighbors of v_k , as is illustrated in Figure 5. We shall denote the neighbor of v_k that is along the path joining v_k and v_1 as w_1 . Note that w_1 must have been infected before v_k when the rumor starts spreading from v_1 . Let w_2, \dots, w_d be the $d - 1$ “children” of v_k , away from v_1 .

For convenience, we shall use notation $t' = t - T_k$ with $t' \geq 0$. Let $\mathcal{T}_i^k(t')$ be the subtree of G rooted at w_i at time t' away from v_k . Therefore, $\mathcal{T}_1^k(t')$ is rooted at w_1 and includes v_1, \dots, v_{k-1} . For $2 \leq i \leq d$, $\mathcal{T}_i^k(t')$ are rooted at w_i and contain nodes in G that are away from v_k . None of the $\mathcal{T}_i^k(t')$ for $1 \leq i \leq d$ include v_k . When v_k is infected at time

Figure 5. (Color online) Illustration of the labeling of the neighbors of v_k and their subtrees for $k = 3$ in a rumor graph at t_k (the time of infection of v_k).



Note. The rumor infected nodes are colored black, and the uninfected nodes are white.

T_k , we have that $T_1^k(0) = k - 1$, and $T_i^k(0) = 0$ for $2 \leq i \leq d$. This notation is illustrated in Figure 5.

By definition $\mathcal{T}_1^k(t')$ is never empty, but $\mathcal{T}_i^k(t')$ can be empty if w_i is not infected, for $2 \leq i \leq d$. As before, let $T_i^k(t') = |\mathcal{T}_i^k(t')|$. As per Lemma 1, v_k is identified as a rumor center if and only if all of its d subtrees are balanced, i.e.,

$$2T_i^k(t') \leq 1 + \sum_{j=1}^d T_j^k(t'), \quad \forall 1 \leq i \leq d. \quad (15)$$

Therefore, for $t \geq T_k$ with $t' = t - T_k$,

$$\begin{aligned} \mathbf{P}(C_t^k) &\geq \mathbf{P}\left(\bigcap_{i=1}^d E_i\right) = 1 - \mathbf{P}\left(\bigcup_{i=1}^d E_i^c\right) \\ &\geq 1 - \sum_{i=1}^d \mathbf{P}(E_i^c), \quad \text{and} \end{aligned} \quad (16)$$

$$\begin{aligned} \mathbf{P}(C_t^k) &\leq \mathbf{P}\left(\bigcap_{i=1}^d F_i\right) = 1 - \mathbf{P}\left(\bigcup_{i=1}^d F_i^c\right) \\ &= 1 - \sum_{i=1}^d \mathbf{P}(F_i^c), \end{aligned} \quad (17)$$

where $E_i = \{2T_i^k(t') < 1 + \sum_{j=1}^d T_j^k(t')\}$ and $F_i = \{2T_i^k(t') \leq 1 + \sum_{j=1}^d T_j^k(t')\}$.

As before, we shall evaluate these probabilities by studying the evolution of the appropriate rumor boundaries. However,

unlike for $k = 1$, when $k \geq 2$ the rumor boundaries have asymmetric initial conditions. Specifically, $\mathcal{T}_1^k(0) = k - 1$, $Z_1^k(\cdot) = (d - 2)(k - 1) + 1$ and for $2 \leq i \leq d$, $\mathcal{T}_i^k(0) = 0$ and $Z_i^k(0) = 1$. Beyond this difference, the rules governing the evolution of the rumor boundaries are the same as those described in the proof for $k = 1$. To evaluate E_1^c (and F_1^c), we consider a Polya's urn in which we start with $(d - 2)(k - 1) + 1$ balls of type 1 (corresponding to $Z_1^k(0)$) and $d - 1$ balls of type 2 (corresponding to $\sum_{j=2}^d Z_j^k(0)$). With these initial conditions, the limit law of fraction of balls of type 1 turns out to be (see Athreya and Ney 1972 for details) a Beta distribution with parameters $a = ((d - 2)(k - 1) + 1)/(d - 2) = (k - 1) + 1/(d - 2)$ and $b = (d - 1)/(d - 2) = 1 + 1/(d - 2)$. Finally, since the fraction of balls of type 1, i.e., the ratio $Z_1^k(t')/(\sum_{j=1}^d Z_j^k(t'))$, equals $T_1^k(t')/(1 + \sum_{j=1}^d T_j^k(t'))$ as $t' \rightarrow \infty$, we obtain

$$\begin{aligned} \lim_{t' \rightarrow \infty} \mathbf{P}(E_1^c) &= \lim_{t' \rightarrow \infty} \mathbf{P}(F_1^c) \\ &= 1 - I_{1/2}\left(k - 1 + \frac{1}{d - 2}, 1 + \frac{1}{d - 2}\right). \end{aligned} \quad (18)$$

For $2 \leq i \leq d$, in the corresponding Polya's urn model, we start with 1 ball of type 1 and $k(d - 2) + 1$ balls of type 2. Therefore, using an identical sequence of arguments, we obtain that for $2 \leq i \leq d$,

$$\lim_{t' \rightarrow \infty} \mathbf{P}(E_i^c) = \lim_{t' \rightarrow \infty} \mathbf{P}(F_i^c) = 1 - I_{1/2}\left(\frac{1}{d - 2}, k + \frac{1}{d - 2}\right). \quad (19)$$

From (16)–(19), it follows that

$$\begin{aligned} \lim_{t \rightarrow \infty} \mathbf{P}(C_t^k) &= I_{1/2}\left(k - 1 + \frac{1}{d - 2}, 1 + \frac{1}{d - 2}\right) \\ &\quad + (d - 1)\left(I_{1/2}\left(\frac{1}{d - 2}, k + \frac{1}{d - 2}\right) - 1\right). \end{aligned} \quad (20)$$

This establishes (5) for all k and completes the proof of Theorem 3.1.

4.2. Proof of Lemma 1

We provide here a proof of Lemma 1 for the convenience of the reader. Much of this proof is taken from Shah and Zaman (2010). We begin by establishing the following property about rumor centrality.

PROPOSITION 1. Consider an undirected tree graph $G = (V, E)$ with $|V| = N$ and any two neighboring nodes $u, v \in V$ such that $(u, v) \in E$. The rumor centralities of these two nodes satisfy the following relationship:

$$\frac{R(u, G)}{R(v, G)} = \frac{T_u^v}{N - T_u^v}. \quad (21)$$

We now show that if v is a rumor center then it must satisfy the condition given by Equation (11) in Lemma 1.

For any node i neighboring the rumor center v , Proposition 1 gives

$$\frac{R(i, G)}{R(v, G)} = \frac{T_i^v}{N - T_i^v} \leq 1.$$

Rearranging terms, we obtain

$$T_i^v \leq \frac{N}{2} \leq \frac{1}{2} \left(1 + \sum_{j \in \mathcal{N}(v)} T_j^v \right).$$

We now establish the other direction of Lemma 1. Assume Equation (11) of the Lemma is satisfied for a node v . We now show that v must be a rumor center.

Let $i \in V$ be a node d hops from v and let $\{v_0 = v, v_1, v_2, \dots, v_d = i\}$ be the sequence of nodes in the path between v and i . Using Proposition 1 we obtain

$$\frac{R(i, G)}{R(v, G)} = \prod_{i=1}^d \frac{R(v_i, G)}{R(v_{i-1}, G)} = \prod_{i=1}^d \frac{T_{v_i}^{v_{i-1}}}{N - T_{v_i}^{v_{i-1}}}.$$

The subtrees on the path between v and i have the special property that $T_{v_i}^{v_{i-1}} = T_{v_i}^v$ for $i = 1, 2, \dots, d$ because the nodes in the subtree rooted at v_i are the same if the subtree is directed away from v_{i-1} or v . We also have the property that $N/2 \geq T_{v_{i-1}}^v > T_{v_i}^v$ for $i = 2, \dots, d$ because the subtrees must decrease in size by at least one node as we traverse the path from v to i and node v satisfies Equation (11) in the Lemma. With these facts we obtain

$$\frac{R(i, G)}{R(v, G)} = \prod_{i=1}^d \frac{T_{v_i}^v}{N - T_{v_i}^v} \leq 1. \tag{22}$$

If the inequality is strict in Equation (11), then we have that for any $i \neq v$, $T_i^v < N/2$. Using Proposition 1 it can be shown that this implies that for every $i \neq v$, there exists a node $j \neq i$ such that $T_j^i > N/2$. This violates Equation (11), which means i cannot be a rumor center. Therefore, v is the unique rumor center.

4.3. Proof of Proposition 1

The rumor centrality of a node v in a tree $G = (V, E)$ with $|V| = N$ is given by

$$R(v, G) = \frac{N!}{\prod_{w \in V} T_w^v}$$

with the tree variables T_w^v denoting the size of the subtree of G that is rooted at w and points away from v . For any two nodes u, v in a tree such that $(u, v) \in E$ there is a special relationship between their subtrees. For any $w \in V$, $w \neq u, v$, it can be shown that $T_w^v = T_w^u$. Also, it can be shown that \mathcal{T}_u^v contains all nodes which are not in \mathcal{T}_v^u . This gives the simple relation that $T_u^u = N - T_u^v$. With these results on the subtree variables we obtain

$$\frac{R(u, G)}{R(v, G)} = \frac{\prod_{w \in V} T_w^v}{\prod_{w \in V} T_w^u} = \frac{T_u^v}{N - T_u^v}.$$

4.4. Proof of Corollary 1

Simple analysis yields Corollary 1. We start by defining the asymptotic probability for a d -regular tree as $\lim_{t \rightarrow \infty} \mathbf{P}(C_t^1) = \alpha_d$. This quantity then becomes

$$\begin{aligned} \alpha_d &= dI_{1/2} \left(\frac{1}{d-2}, 1 + \frac{1}{d-2} \right) - d + 1 \\ &= 1 - \frac{d\Gamma(1 + 2/(d-2))}{\Gamma(1/(d-2))\Gamma(1 + 1/(d-2))} \\ &\quad \cdot \int_{1/2}^1 t^{1/(d-2)-1} (1-t)^{1/(d-2)} dt \end{aligned}$$

We then take the limit as d approaches infinity.

$$\begin{aligned} \lim_{d \rightarrow \infty} \alpha_d &= \lim_{d \rightarrow \infty} 1 - \frac{d\Gamma(1 + 2/(d-2))}{\Gamma(1/(d-2))\Gamma(1 + 1/(d-2))} \\ &\quad \cdot \int_{1/2}^1 t^{1/(d-2)-1} (1-t)^{1/(d-2)} dt \\ &= 1 - \lim_{d \rightarrow \infty} \frac{d\Gamma(1 + 2/(d-2))}{(d-2 - \gamma + O(d^{-1}))\Gamma(1 + 1/(d-2))} \\ &\quad \cdot \int_{1/2}^1 t^{1/(d-2)-1} (1-t)^{1/(d-2)} dt \\ &= 1 - \int_{1/2}^1 t^{-1} dt \\ &= 1 - \ln 2. \end{aligned}$$

Above, γ is the Euler-Mascheroni constant and we have used the following approximation of $\Gamma(x)$ for small x : $\Gamma(x) = x^{-1} - \gamma + O(x)$.

4.5. Proof of Corollary 2

Corollary 2 follows from (5) and monotonicity of the Γ function over $[1, \infty)$. For $k \geq 2$,

$$\begin{aligned} \lim_{t \rightarrow \infty} \mathbf{P}(C_t^k) &= I_{1/2} \left(k-1 + \frac{1}{d-2}, 1 + \frac{1}{d-2} \right) \\ &\quad + (d-1) \left(I_{1/2} \left(\frac{1}{d-2}, k + \frac{1}{d-2} \right) - 1 \right) \\ &\leq I_{1/2} \left(k-1 + \frac{1}{d-2}, 1 + \frac{1}{d-2} \right) \\ &= \frac{\Gamma(k + 2/(d-2))}{\Gamma(k-1 + 1/(d-2))\Gamma(1 + 1/(d-2))} \\ &\quad \cdot \int_0^{1/2} t^{k+1/(d-2)-2} (1-t)^{1/(d-2)} dt \\ &\stackrel{(a)}{\leq} \frac{\Gamma(k + 2/(d-2))}{\Gamma(k-1 + 1/(d-2))\Gamma(1 + 1/(d-2))} \\ &\quad \cdot \int_0^{1/2} t^{k-2} dt \\ &\stackrel{(b)}{\leq} \frac{4e^2\Gamma(k+2)}{\Gamma(k-1)} \int_0^{1/2} t^{k-2} dt \end{aligned}$$

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$$\stackrel{(c)}{\leq} 4e^2 k(k+1)(k+2) \left(\frac{1}{2}\right)^{k-1} \\ \asymp \exp(-\Theta(k)).$$

In above, (a) follows from the fact that $t < 1$ and hence $t^{k-2+1/(d-2)} \leq t^{k-2}$. For (b), we use the following well-known properties of the Γ function: (i) over $[2, \infty)$, the Γ function is nondecreasing and hence for $k \geq 2$ and $d \geq 3$, $\Gamma(k+2/(d-2)) \leq \Gamma(k+2)$; (ii) over $(0, \infty)$, the Γ function achieves its minimal value in $[1, 2]$ which is at least $1/(2e)$ and therefore, along with (i), we have that $\Gamma(k-1+1/(d-2)) \geq \Gamma(k-1)/(2e)$ and $\Gamma(1+1/(d-2)) \geq 1/(2e)$. For (c), we use the fact that $\Gamma(x+1) = x\Gamma(x)$ for any $x \in (0, \infty)$.

4.6. Proof of Theorem 3.2: Correct Detection for Random Trees

The goal is to establish that there is a strictly positive probability of detecting the source correctly as the rumor center when the rumor starts at the root of a generic random tree with generic spreading time distribution as defined earlier. The probability is with respect to the joint distribution induced by the tree construction and the SI rumor spreading model with independent spreading times. We extend the technique employed in the proof of Theorem 3.1. However, it requires using a generalized Polya’s urn or age-dependent branching process as well as delicate technical arguments.

4.6.1. Background: Age-Dependent Branching Process.

We recall a generalization of the classical Polya’s urn known as an age-dependent branching process. Such a process starts at time $t = 0$ with a given finite number of nodes, say $B(0) \geq 1$. Each node remains alive for an independent, identically distributed lifetime with cumulative distribution function given by $F: [0, \infty) \rightarrow [0, 1]$. The lifetime distribution function F will be assumed to be nonatomic at 0, i.e., $F(0^+) = 0$. Each node dies after remaining alive for its lifetime. Upon the death of a node, it gives birth to random number of nodes, say η . The random variables η corresponding to each node are independent and identically distributed over the non-negative integers. The newly born nodes live for their lifetime and the upon death give birth to new nodes, and so on.

As can be seen, the classical Galton-Watson process is a special case of this general model and the size of the entire urn in the Polya’s urn process described earlier naturally fits this model. An interested reader is referred to Athreya and Ney (1972) for a detailed exposition. Next, we recall certain remarkable asymptotic properties of this process that will be crucially utilized. We start with a useful definition.

DEFINITION 4 (ATHREYA AND NEY 1972, p. 146). Let $m \equiv \mathbb{E}[\eta]$. The Malthusian parameter $\alpha = \alpha(m, F)$ of an age-dependent branching process is the unique solution, if it exists, of the equation

$$m \int_0^\infty e^{-\alpha y} dF(y) = 1. \tag{23}$$

A sufficient condition for the existence of the Malthusian parameter is $m = \mathbb{E}[\eta] > 1$. As an example, consider process where spreading time distribution is exponential with parameter λ , i.e., $F(t) = 1 - e^{-\lambda t}$, and let $m = \mathbb{E}[\eta] > 1$. The Malthusian parameter $\alpha(m, F)$ is given by the solution of

$$m \int_0^\infty e^{-\alpha y} \lambda e^{-\lambda y} dy = 1,$$

which is

$$\alpha(m, F) = \lambda(m - 1).$$

The Malthusian parameter captures the average growth rate of the branching process. We now recall the following result.

THEOREM 4.2 (ATHREYA AND NEY 1972, THEOREM 2, p. 172). Consider an age-dependent branching process as described above with the additional properties that $m = \mathbb{E}[\eta] > 1$ and $\mathbb{E}[\eta \log \eta] < \infty$. Let $\alpha \equiv \alpha(m, F)$ be the Malthusian parameter of the process and define

$$c = \frac{m - 1}{\alpha m^2 \int_0^\infty y e^{-\alpha y} dF(y)}.$$

Let $B(t)$ denote the number of nodes alive in the process at time $t \geq 0$. Then

$$\frac{1}{c e^{\alpha t}} B(t) \xrightarrow{t \rightarrow \infty} W \quad \text{in distribution,}$$

where W is such that

$$\mathbb{E}[W] = 1 \tag{24}$$

$$\mathbf{P}(W = 0) = q, \tag{25}$$

$$\mathbf{P}(W \in (x_1, x_2)) = \int_{x_1}^{x_2} w(y) dy, \quad \text{for } 0 < x_1 < x_2 < \infty, \tag{26}$$

where $q \in (0, 1)$ is the smallest root of the equation $\sum_{k=0}^\infty s^k \mathbf{P}(\eta = k) = s$ and $w(\cdot)$ is absolutely continuous with respect to the Lebesgue measure so that $\int_0^\infty w(y) dy = 1 - q$.

The above result states that with probability q ($0 < q < 1$) the branching process becomes extinct, and with probability $1 - q$ the size of the process scales as $\exp(\alpha t)$ for large t . We will need finer control on the asymptotic growth of the branching process. Precisely, we shall use the following implication of the above stated result.

COROLLARY 3. Under the setting of Theorem 4.2, for any $f > 1$, there exists an $x > 0$, so that

$$\mathbf{P}(W \in (x, fx)) > 0. \tag{27}$$

PROOF. Define

$$a_k = f^k, \quad \text{for } k \in \mathbb{Z}.$$

By definition, $\{W > 0\} = \bigcup_{k \in \mathbb{Z}} \{W \in (a_k, a_{k+1}]\}$. Because of the absolute continuity of $w(\cdot)$ in (26), it follows that $\mathbf{P}(W = a_k) = 0$ for all $k \in \mathbb{Z}$. Therefore, it follows that

$$0 < \mathbf{P}(W > 0) \\ = \mathbf{P}\left(\bigcup_{k \in \mathbb{Z}} \{W \in (a_k, a_{k+1})\}\right) \\ \leq \sum_{k \in \mathbb{Z}} \mathbf{P}(W \in (a_k, a_{k+1})). \tag{28}$$

From above, it follows that there exists a k such that $\mathbf{P}(W \in (a_k, a_{k+1})) > 0$. This completes the proof.

4.6.2. Notation. We quickly recall some notation. To start with, as before let v_1 be the root node of the tree. It has η_0 children distributed as per \mathcal{D}_0 . Define the event $A = \{\eta_0 \geq 3\}$. By assumption of Theorem 3.2, $\mathbf{P}(A) > 0$. We shall show that

$$\liminf_{t \rightarrow \infty} \mathbf{P}(C_t^1 | A) > 0, \tag{29}$$

as it will imply the desired result $\liminf_{t \rightarrow \infty} \mathbf{P}(C_t^1) > 0$, using $\mathbf{P}(A) > 0$ since $\mathbf{P}(C_t^1) \geq \mathbf{P}(C_t^1 | A)\mathbf{P}(A)$. Therefore, we shall consider conditioning on event A and let $d = \eta_0 \geq 3$ for remainder of the proof. Note that all the spreading times as well as all other randomness are independent of η_0 . The only effect of conditioning on A is that we know that root has $d \geq 3$ children. Let u_1, \dots, u_d be the d children of root v_1 . The random tree \mathcal{G} is constructed by adding a random number of children to u_1, \dots, u_d recursively as per distribution \mathcal{D} as explained in §3.2.

Furthermore, as explained in §3.2, the rumor spreads on \mathcal{G} starting from v_1 at time 0 as per the spreading times with cumulative distribution function F that is nonatomic at 0. Let G be the sub-tree of \mathcal{G} that is infected at time t with $n(t)$ infected nodes in G at time t . Let $\mathcal{T}_i(t)$ denote the subtree of G rooted at node u_i (pointing away from root v_1) at time t , for $1 \leq i \leq d$ and let $T_i(t) = |\mathcal{T}_i(t)|$. By definition $T_i(0) = 0$ for $1 \leq i \leq d$. Let $Z_i(t)$ denote the size of the rumor boundary of $\mathcal{T}_i(t)$; initially $Z_i(0) = 1$, $1 \leq i \leq d$.

Now let us consider the evolution of $Z_i(\cdot)$: recall that each node in the rumor boundary has a rumor infected parent (neighbor). This node will become infected after the amount of time given by the spreading time associated with the edge connecting the node with its infected parent. After the node becomes infected, it is no longer part of the rumor boundary, but all of its uninfected neighbors (children) become part of the rumor boundary. And as per the random generative process of the tree construction, the number of children added, η , has distribution \mathcal{D} . Therefore, the rumor boundary process $Z_i(\cdot)$ for each $1 \leq i \leq d$ is exactly an age-dependent branching process. Furthermore, each $Z_i(\cdot)$ evolves independently and since initially each starts at the same time with exactly one node, they are identically distributed. Therefore, we can utilize the results stated in §4.6.1 to characterize the properties of $Z_i(\cdot)$ for $1 \leq i \leq d$. In the case of regular trees, $Z_i(\cdot)$ and $T_i(\cdot)$ were linearly related which allowed us to obtain results about $T_i(\cdot)$ and the desired conclusion. While in this general setting, $Z_i(\cdot)$ and $T_i(\cdot)$ are not linearly related, we show that they are asymptotically linearly related because of an appropriate Law of Large Numbers effect. This will help us obtain the desired conclusion. We present the details next.

4.6.3. Correct Detection. As before, we wish to show that

$$\mathbf{P}(C_t^1 | A) \geq \mathbf{P}\left(\bigcap_{i=1}^d \left\{2T_i(t) < 1 + \sum_{j=1}^d T_j(t)\right\}\right), \tag{30}$$

where we have removed the conditioning on A , as the only effect of A was having d distinct trees, which is already captured. We shall establish (30) in two steps:

Step 1. Using the characterizations of $Z_i(\cdot)$ in terms of age dependent branching processes as discussed above, we shall show that there is a nontrivial event $\mathcal{E}_1 \subset \bigcap_{i=1}^d \{2Z_i(t) < \sum_{j=1}^d Z_j(t)\}$ with $\liminf_{t \rightarrow \infty} \mathbf{P}(\mathcal{E}_1) > 0$.

Step 2. Identify an event $\mathcal{E}_2 \subset \mathcal{E}_1$ with $\liminf_{t \rightarrow \infty} \mathbf{P}(\mathcal{E}_2) > 0$ and $\mathcal{E}_2 \subset \bigcap_{i=1}^d \{2T_i(t) < 1 + \sum_{j=1}^d T_j(t)\}$ for all t large enough.

This will yield the desired results.

4.6.4. Step 1. For any $x > 0$ and $\varepsilon > 0$ define the event $\mathcal{E}(x, \varepsilon, t)$ as

$$\begin{aligned} \mathcal{E}(x, \varepsilon, t) &= \bigcap_{i=1}^d \{Z_i(t)c^{-1}e^{-\alpha t} \in (x, (1-3\varepsilon)(d-1)x)\}. \end{aligned} \tag{31}$$

Since $d \geq 3$, $(1-3\varepsilon)(d-1) > 1$ for small enough $\varepsilon > 0$ and hence the above event is well defined. It can be easily checked that $\mathcal{E}(x, \varepsilon, t) \subset \bigcap_{i=1}^d \{2Z_i(t) < \sum_{j=1}^d Z_j(t)\}$, since under this event,

$$\max_i Z_i(t) \leq (1-3\varepsilon)(d-1)x < (d-1)x \leq (d-1)\min_i Z_i(t).$$

By Theorem 4.2, it follows that $Z_i(t)c^{-1}e^{-\alpha t}$ converges to W_i , which are independent across i and identically distributed as per (25)–(26). Therefore, using Corollary 3, it follows that there exists an $x^* > 0$ such that

$$\liminf_{t \rightarrow \infty} \mathbf{P}(\mathcal{E}(x^*, \varepsilon, t)) > 0. \tag{32}$$

Define $\mathcal{E}_1 = \mathcal{E}(x^*, \varepsilon, t)$.

4.6.5. Step 2. We want to find $\mathcal{E}_2 \subset \mathcal{E}_1$ so that for t large enough, $\mathcal{E}_2 \subset \bigcap_{i=1}^d \{2T_i(t) < 1 + \sum_{j=1}^d T_j(t)\}$ and $\liminf_{t \rightarrow \infty} \mathbf{P}(\mathcal{E}_2) > 0$. For regular trees this was achieved by using the linear (deterministic) relationship between the $Z_i(\cdot)$ and $T_i(\cdot)$. Here, we do not have such a relationship. Instead, we shall establish an asymptotic relationship. To that end, recall that for any $t \geq 0$,

$$Z_i(t) = 1 + \sum_{\ell \in \mathcal{T}_i(t)} (\eta_\ell - 1). \tag{33}$$

The above holds because as per the branching process, when a node in the “boundary” dies (-1 is added to $Z_i(\cdot)$) and it is added to $T_i(\cdot)$, η_ℓ new nodes are added to boundary.

Consider $T_i(\cdot)$. It grows by adding nodes with a random number of children as per distribution \mathcal{D} independently. Let η_1, η_2, \dots be these random number of children added to it in that order (we assume this sequence to be infinite irrespective of whether or not $T_i(\cdot)$ stops growing). Since these are i.i.d. random variables with finite mean (actually, $\mathbf{E}[\eta \log \eta] < \infty$), by the standard Strong Law of Large Numbers, for any

small enough $\varepsilon, \delta > 0$, with probability at least $1 - \delta$, for all $1 \leq i \leq d$, we have that for all $p \geq 1$

$$\frac{(1 - \varepsilon)p}{m} - C(\varepsilon, \delta) \leq N_i(p) \leq \frac{(1 + \varepsilon)p}{m} + C(\varepsilon, \delta) \quad (34)$$

where $N_i(p) = \inf\{\ell: \sum_{j=1}^{\ell} (\eta_j - 1) \geq p\}$, $m = \mathbf{E}[\eta]$ and $C(\varepsilon, \delta)$ is a non-negative constant depending upon ε, δ but independent of p . Let us call the event represented by (34) as $\mathcal{E}'(\varepsilon, \delta)$. Here, we have the freedom of choosing as small a δ and ε as we like. We will choose δ so that it is much smaller than the probability of event \mathcal{E}_1 for t large enough. Given such a choice, it will follow that for all t large enough, the event $\mathcal{E}_2 = \mathcal{E}_1 \cap \mathcal{E}'(\varepsilon, \delta)$ has strictly positive probability. Under event \mathcal{E}_2 , we have (with the definition $\hat{Z}_i(t) = Z_i(t)c^{-1}e^{-at}$)

$$\begin{aligned} \hat{Z}_i(t) &\in (x^*, x^*(1 - 3\varepsilon)(d - 1)), \quad \text{for all } 1 \leq i \leq d, \\ T_i(t)c^{-1}e^{-at} &\in \left(\frac{\hat{Z}_i(t)(1 - \varepsilon)}{m} - a_t, \frac{\hat{Z}_i(t)(1 + \varepsilon)}{m} + a_t \right) \quad (35) \\ &\text{for all } 1 \leq i \leq d, \end{aligned}$$

where the constants $a_t \rightarrow 0$ as $t \rightarrow \infty$. Therefore, it can be easily checked that for t large enough and ε small enough, $\mathcal{E}_2 \subset \bigcap_{i=1}^d \{2T_i(t) < 1 + \sum_{j=1}^d T_j(t)\}$, just the way we argued that $\mathcal{E}_1 \subset \bigcap_{i=1}^d \{2Z_i(t) < \sum_{j=1}^d Z_j(t)\}$. As discussed above, with an appropriate choice of δ and ε , we can guarantee that $\liminf_{t \rightarrow \infty} \mathbf{P}(\mathcal{E}_2) > 0$. This concludes the search for the desired event \mathcal{E}_2 and we have established the desired claim of $\liminf_{t \rightarrow \infty} \mathbf{P}(C_t^1) > 0$. This completes the proof of Theorem 3.2.

4.7. Proof of Theorem 3.3

4.7.1. Background: Properties of Age-Dependent Branching Processes. We shall utilize the following property known in the literature about bounds on the moment generating function of the size of an age-dependent branching process. We shall assume the notation from the earlier section.

THEOREM 4.3 (NAKAYAMA ET AL. 2004, THEOREM 3.1). Consider an age dependent branching process with the properties that $m = \mathbf{E}[\eta] > 1$, $\mathbf{E}[\exp(\theta\eta)] < \infty$ for all $\theta \in (0, \theta_1)$ for some $\theta_1 > 0$, and the spreading time distribution is nonatomic. Let $B(t)$ represent the number of living nodes in the branching process at time t and let $V(t)$ represent the number of nodes born before time t . Then, there exists a $\theta^* > 0$ such that for all $\theta \in (-\theta^*, \theta^*)$

$$\mathbf{E}[e^{\theta B(t)}] \leq \mathbf{E}[e^{\theta V(t)}] < \infty. \quad (36)$$

4.7.2. Background: Two Inequalities. We state two useful concentration-style inequalities that we shall derive here for completeness.

PROPOSITION 2. For $i \geq 1$ let X_i be independent and identically distributed random variables such that $\mathbf{E}[\exp(\theta X_1)] < \infty$ for all $\theta \in (-\delta, \delta)$ for some $\delta > 0$. Then, for any $\varepsilon > 0$, there exists constants $C_1, C_2(\varepsilon, \delta) > 0$ such that

$$\mathbf{P}\left(\sum_{i=1}^n X_i \leq \mu n(1 - \varepsilon)\right) \leq C_1 \exp(-C_2(\varepsilon, \delta)\mu n), \quad (37)$$

where $\mu = \mathbf{E}[X_1]$.

PROPOSITION 3. Consider independent and identical random variables X_1, \dots, X_{r+s} for integers r, s such that $1 \leq s < r$. Let $\mu = \mathbf{E}[X_1]$ and $\mathbf{E}[\exp(\theta X_1)] < \infty$ for all $\theta \in (-\delta, \delta)$ for some $\delta > 0$. Then there exists a constant c such that for any $\gamma > 0$, there exists a constant $\theta^* = \min((\gamma + (r - s)\mu)/(2(r + s)c), \delta_1/2)$ for some $0 < \delta_1 < \delta$, such that

$$\mathbf{P}\left(\sum_{i=1}^r X_i - \sum_{j=1}^s X_{r+j} \leq -\gamma\right) \leq \exp\left(-\frac{1}{2}\theta^*(\gamma + (r - s)\mu)\right). \quad (38)$$

Next, we prove these two propositions.

PROOF OF PROPOSITION 2. Let X be a random variable with identical distribution as that of $X_i, i \geq 1$. By assumption in the Proposition statement, it follows that for $\theta \in (-\delta, \delta)$

$$\begin{aligned} M_X(\theta) &\equiv \log \mathbf{E}[\exp(\theta X)] \\ &= \log\left(1 + \sum_{j=1}^{\infty} \frac{\theta^j \mathbf{E}[X^j]}{j!}\right) \\ &\leq \log(1 + \theta\mu + c\theta^2), \end{aligned}$$

for some $c > 0$ for all $\theta \in (-\delta_1, \delta_1)$ for some $0 < \delta_1 < \delta$. Using the inequality $\log(1 + x) \leq x$ for all $x > -1$, we obtain

$$M_X(\theta) \leq \theta\mu + c\theta^2. \quad (39)$$

Now, for any $\Gamma > 0$ and $\theta > 0$, using standard arguments and (39), we obtain

$$\begin{aligned} \mathbf{P}\left(\sum_{i=1}^n X_i \leq n\mu - \Gamma\right) &= \mathbf{P}\left(\exp\left(-\theta\left(\sum_{i=1}^n X_i - n\mu\right)\right) \geq \exp(\Gamma\theta)\right) \\ &\leq \exp(-\theta\Gamma + \theta n\mu) \mathbf{E}[\exp(-\theta X)]^n \\ &= \exp(-\theta\Gamma + \theta n\mu + nM_X(-\theta)) \\ &\leq \exp(-\theta\Gamma + cn\theta^2) \quad (40) \end{aligned}$$

For any $0 < \theta \leq \Gamma/(2nc)$,

$$\mathbf{P}\left(\sum_{i=1}^n X_i \leq n\mu - \Gamma\right) \leq \exp\left(-\frac{1}{2}\Gamma\theta\right).$$

Using $\Gamma = n\mu\varepsilon$ and $\theta^* = \min(\delta/2, \mu\varepsilon/(2c))$, we have

$$\begin{aligned} \mathbf{P}\left(\sum_{i=1}^n X_i \leq n\mu(1-\varepsilon)\right) &\leq \exp\left(-\frac{1}{2}n\mu\varepsilon\theta^*\right) \\ &= \exp\left(-C_2(\varepsilon, \delta)n\mu\right), \end{aligned} \tag{41}$$

where $C_2(\varepsilon, \delta) = \frac{1}{2}\varepsilon \min(\delta/2, \mu\varepsilon/(2c))$. This completes the proof of Proposition 2.

PROOF OF PROPOSITION 3. Given $1 \leq s < r$, $\gamma > 0$ and $\theta > 0$, using standard arguments (with the notation that the random variable X has an identical distribution as X_i , $1 \leq i \leq r+s$)

$$\begin{aligned} \mathbf{P}\left(\sum_{i=1}^r X_i - \sum_{j=1}^s X_{r+j} \leq -\gamma\right) &= \mathbf{P}\left(-\theta\left(\sum_{i=1}^r X_i - \sum_{j=1}^s X_{r+j}\right) \geq \gamma\theta\right) \\ &\leq \exp(-\theta\gamma)\mathbf{E}[\exp(-\theta X)]^r \mathbf{E}[\exp(\theta X)]^s. \end{aligned}$$

Using notation and arguments similar to that in the proof of Proposition 2, we conclude that the above inequality can be bounded above, for some $c > 0$ and $\theta \in (-\delta_1, \delta_1)$ for $0 < \delta_1 < \delta$ as

$$\begin{aligned} \mathbf{P}\left(\sum_{i=1}^r X_i - \sum_{j=1}^s X_{r+j} \leq -\gamma\right) &\leq \exp(-\theta\gamma + (s-r)\theta\mu + (r+s)c\theta^2). \end{aligned} \tag{42}$$

For $\theta^* = \min((\gamma + (r-s)\mu)/(2(r+s)c), \delta_1/2)$, we obtain

$$\begin{aligned} \mathbf{P}\left(\sum_{i=1}^r X_i - \sum_{j=1}^s X_{r+j} \leq -\gamma\right) &\leq \exp\left(-\frac{1}{2}\theta^*(\gamma + (r-s)\mu)\right). \end{aligned} \tag{43}$$

4.7.3. Proof of Theorem 3.3. Theorem 3.3 assumes that the spreading times have an exponential distribution with (unknown) parameter $\lambda > 0$ for all edges. The underlying graph is a generic random tree, just like that in Theorem 3.2. We shall crucially utilize the “memory-less” property of the exponential distribution to obtain the exponential error bound on $\limsup_{t \rightarrow \infty} \mathbf{P}(C_t^k)$ claimed in Theorem 3.3.

To that end, continuing with notations from the proof of Theorem 3.2, let $T_k = \inf\{t > 0: T(t) = k\}$. By definition,

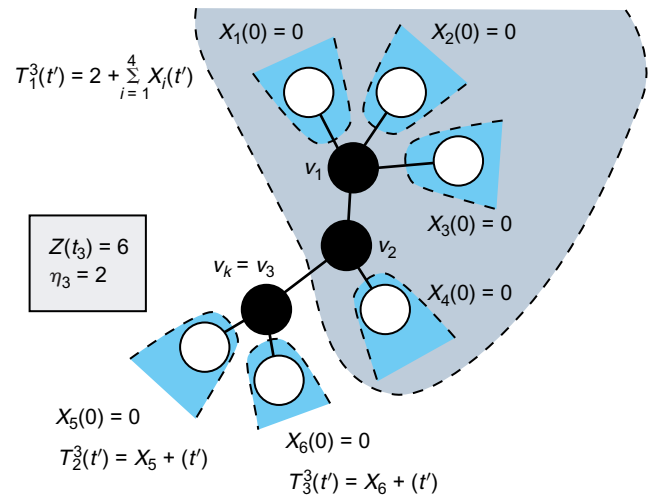
$$\limsup_{t \rightarrow \infty} \mathbf{P}(C_t^k | T_k = \infty) = 0. \tag{44}$$

Therefore,

$$\limsup_{t \rightarrow \infty} \mathbf{P}(C_t^k) \leq \limsup_{t \rightarrow \infty} \mathbf{P}(C_t^k | T_k < \infty). \tag{45}$$

Therefore, let us assume that $T_k < \infty$ and we will be interested in $t > T_k$. We shall redefine the index for time as

Figure 6. (Color online) Illustration of the labeling of the subtree random processes $X_j(t')$ for $k = 3$ in a rumor graph at $t = t_k$ (the time of infection of v_k).



Note. The rumor infected nodes are colored black, and the uninfected nodes are white.

$t' = t - T_k$. When $t' = 0$, we have exactly k nodes infected and let them be v_1, \dots, v_k , chronologically infected in that order. Let $d = \eta_k + 1$ denote the total number of neighbors of v_k , let w_1 denote the neighbor of v_k on the path connecting v_k and v_1 and let w_2, \dots, w_d be the other neighbors of v_k . Let $\mathcal{T}_1^k(t')$ (with $T_1^k(t') = |\mathcal{T}_1^k(t')|$) be the sub-tree rooted at w_1 including v_1 (and not including v_k). Similarly, let $\mathcal{T}_j^k(t')$ (with $T_j^k(t') = |\mathcal{T}_j^k(t')|$) be the sub-tree rooted at w_j , not including v_k for $2 \leq j \leq d$. By definition $T_1^k(0) = k - 1$ and $T_j^k(0) = 0, 2 \leq j \leq d$.

Let $Z(t')$ be the size of the rumor boundary of the graph at t' and let $\zeta_k = Z(0)$ be the size of the rumor boundary immediately after the k th node is infected. By definition, $\zeta_k \geq d - 1$ as w_2, \dots, w_d are part of the rumor boundary when $t' = 0$. Let $X_1(t'), \dots, X_{\zeta_k}(t')$ be the size of the sub-trees at time $t' = t - T_k$ (for $t \geq T_k$) growing from these ζ_k rumor boundary nodes. This is illustrated in Figure 6. Because of the memory-less property of exponential spreading time distribution, it can be argued that $X_1(t'), \dots, X_{\zeta_k}(t')$ are independent and identically distributed random variables. Putting the above discussion together, we have that

$$T_1^k(t') = k - 1 + \sum_{j=1}^{\zeta_k - d + 1} X_j(t'), \tag{46}$$

$$\sum_{i=2}^d T_i^k(t') = \sum_{j=\zeta_k - d + 2}^{\zeta_k} X_j(t').$$

With $T_k < \infty$, for $t \geq T_k$ and $t' = t - T_k$,

$$\begin{aligned} \mathbf{P}(C_t^k | T_k < \infty) &\leq \mathbf{P}\left(\bigcap_{i=1}^d \left\{2T_i^k(t') \leq \sum_{j=1}^d T_j^k(t')\right\} + 1 \mid T_k < \infty\right) \end{aligned}$$

$$\begin{aligned} &\leq \mathbf{P}\left(T_1^k(t') \leq \sum_{j=2}^d T_j^k(t') + 1 \mid T_k < \infty\right) \\ &\leq \mathbf{P}\left(k - 1 + \sum_{j=1}^{\zeta_k - d + 1} X_j(t') \leq \sum_{j'=\zeta_k - d + 2}^{\zeta_k} X_{j'}(t') + 1\right). \end{aligned} \quad (47)$$

We shall argue that the term on the right in (47) is bounded from above by $O(\exp(-ak))$ for all k large enough. To that end, we shall utilize Propositions 2 and 3.

First, recall that $\zeta_k - d + 1$, which is the total number of nodes in the rumor boundary at time $t = T_k$ because of the first $k - 1$ infected nodes, equals $\sum_{i=1}^{k-1} (\eta_i - 1)$, where $\eta_1, \dots, \eta_{k-1}$ are the random number of children of the first $k - 1$ infected nodes. By assumption, $\mathbf{E}[\eta] > 1$. Therefore, using Proposition 2, it follows that for an appropriate choice of constants C_1, C_2 ,

$$\mathbf{P}(\zeta_k - d + 1 \leq (k - 1)(\mathbf{E}[\eta] - 1)/2) \leq C_1 \exp(-C_2 k). \quad (48)$$

Second, consider the rumor boundary induced because of the children of v_k , denoted in the above sum as $d - 1$ nodes (corresponding to the terms in the right hand side of the equation). Since d , the degree of v_k is a random number distributed as per η and $\mathbf{E}[\exp(\theta\eta)] < \infty$ for all $\theta \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$, it follows that for appropriate constants $C_3, C_4 > 0$ (with $k \geq 2$),

$$\mathbf{P}(d - 1 > (k - 1)(\mathbf{E}[\eta] - 1)/4) \leq C_3 \exp(-C_4 k). \quad (49)$$

Define the event $E = \{\zeta_k - d + 1 > (k - 1)(\mathbf{E}[\eta] - 1)/2\} \cap \{d - 1 \leq (k - 1)(\mathbf{E}[\eta] - 1)/4\}$. Then, from (48)–(49), we have $\mathbf{P}(E^c) \leq C_5 \exp(-C_6 k)$ where $C_5 = C_1 + C_3$ and $C_6 = \min(C_2, C_4)$.

Finally, to bound $\mathbf{P}(F)$, where $F = \{k - 1 + \sum_{j=1}^{\zeta_k - d + 1} X_j(t') \leq \sum_{j'=\zeta_k - d + 2}^{\zeta_k} X_{j'}(t') + 1\}$, consider the following: for all k large enough, using Proposition 3, we have

$$\begin{aligned} \mathbf{P}(F) &\leq \mathbf{P}(F \mid E) + \mathbf{P}(E^c) \\ &\leq C_7 \exp(-C_8 k) + C_5 \exp(-C_6 k) \\ &= C' \exp(-C'' k). \end{aligned} \quad (50)$$

In the last inequality, the first term is derived by applying Proposition 3 where $r \geq (k - 1)(\mathbf{E}[\eta] - 1)/2$ and $s \leq (k - 1)(\mathbf{E}[\eta] - 1)/4$, i.e., $r \geq 2s$, $\gamma = k - 2$, and $C', C'' > 0$ are appropriate constants depending upon C_5, C_6, C_7 and C_8 . Note that the conditions of Proposition 3 are satisfied because of Theorem 4.3. This completes the proof of Theorem 3.3.

4.8. Proof of Theorem 3.4: Geometric Trees

The proof of Theorem 3.4 uses the characterization of the rumor center provided by Proposition 1. That is, we wish to show that for all n large enough, the event that the size of the d^* rumor infected sub-trees of the source v^* are essentially “balanced” occurs with high probability. To establish this, we

shall use coarse estimations on the size of each of these sub-trees using the standard concentration property of renewal processes along with geometric growth. This will be unlike the proof for regular trees where we had to necessarily delve into very fine detailed probabilistic estimates of the size of the sub-trees to establish the result. This relatively easier proof for geometric trees (despite their heterogeneity) brings out the fact that it is fundamentally much more difficult to analyze expanding trees than geometric structures as expanding trees do not yield to generic concentration based estimations as they necessarily have very high variances.

To that end, we shall start by obtaining sharp estimates on the size of each of the d^* rumor infected sub-trees of v^* for any given time t . We are assuming here that the spreading times have a distribution F with mean $\mu > 0$ and an exponential tail (precisely, if X is random variable with distribution F , then $\mathbf{E}[\exp(\theta X)] < \infty$ for $\theta \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$). Initially, at time 0 the source node v^* is infected with the rumor. It starts spreading to its d^* children (neighbors). Let $T_i(t)$ denote the size of the rumor infected subtree, denoted by $\mathcal{F}_i(t)$, rooted at the i th child (or neighbor) of node v^* . Initially, $T_i(0) = 0$. Because of the balanced and geometric growth conditions assumed in Theorem 3.4, the following will be satisfied: for small enough $\varepsilon > 0$ (a) every node within a distance $(t/\mu)(1 - \varepsilon)$ of v^* is in one of the $\mathcal{F}_i(t)$, and (b) no node beyond distance $(t/\mu)(1 + \varepsilon)$ of v^* is in any of the $\mathcal{F}_i(t)$. Such a tight characterization of the “shape” of $\mathcal{F}_i(t)$ along with the polynomial growth will provide sharp enough bound on $T_i(t)$ that will result in establishing Theorem 3.4. This result is summarized below with its proof in §4.8.1.

PROPOSITION 4. Consider a geometric tree with parameters $\alpha > 0$ and $0 < b \leq c$ as assumed in Theorem 3.4 and let the rumor spread from source v^* starting at time 0 as per the SI model with spreading time distribution F such that the mean is μ and $\mathbf{E}[\exp(\theta X)] < \infty$ for $\theta \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$ where X is distributed as per F . Define $\epsilon = t^{-1/2+\delta}$ for any $0 < \delta < 1/2$. Let $G(t)$ be the rumor infected tree at time t . Let \mathcal{G}_t be the set of all trees rooted at v^* (rumor graphs) such that all nodes within distance $(t/\mu)(1 - \epsilon)$ from v^* are in the tree and no node beyond distance $(t/\mu)(1 + \epsilon)$ from v^* is in the tree. Then

$$\mathbf{P}(G_t \in \mathcal{G}_t) = 1 - O(e^{-t^\delta}) \xrightarrow{t \rightarrow \infty} 1.$$

Define \mathcal{E}_t as the event that $G_t \in \mathcal{G}_t$. Under event \mathcal{E}_t , consider the sizes of the sub-trees $T_i(t)$ for $1 \leq i \leq d_{v^*}$. Because of the polynomial growth condition and \mathcal{E}_t , we obtain the following bounds on each $T_i(t)$ for all $1 \leq i \leq d_{v^*}$:

$$\sum_{r=1}^{(t/\mu)(1-\epsilon)-1} br^\alpha \leq T_i(t) \leq \sum_{r=1}^{(t/\mu)(1+\epsilon)-1} cr^\alpha.$$

Now bounding the summations by Riemann integrals, we have

$$\int_0^{L-1} r^\alpha dr \leq \sum_{r=1}^L r^\alpha \leq \int_0^{L+1} r^\alpha dr.$$

Therefore, it follows that under event \mathcal{E}_i , for all $1 \leq i \leq d_{v^*}$

$$\frac{b}{1+\alpha} \left(\frac{t}{\mu} (1-\epsilon) - 2 \right)^{\alpha+1} \leq T_i(t) \leq \frac{c}{1+\alpha} \left(\frac{t}{\mu} (1+\epsilon) \right)^{\alpha+1}.$$

In the most “unbalanced” situation, $d_{v^*} - 1$ of these sub-trees have minimal size $T_{\min}(t)$ and the remaining one sub-tree has size $T_{\max}(t)$ where

$$T_{\min}(t) = \frac{b}{1+\alpha} \left(\frac{t}{\mu} (1-\epsilon) - 2 \right)^{\alpha+1},$$

$$T_{\max}(t) = \frac{c}{1+\alpha} \left(\frac{t}{\mu} (1+\epsilon) \right)^{\alpha+1}.$$

Since by assumption $c < b(d_{v^*} - 1)$, there exists $\gamma > 0$ such that $(1+\gamma)c < b(d_{v^*} - 1)$. Therefore, for any choice of $\epsilon = t^{-1/2+\delta}$ for some $\delta \in (0, 1/2)$, we have

$$\begin{aligned} \frac{(d^* - 1)T_{\min}(t) + 1}{T_{\max}(t)} &= \frac{b(d_{v^*} - 1)}{c} \left(\frac{t/\mu - t^{1/2+\delta} - 2}{t/\mu + t^{1/2+\delta}} \right)^{\alpha+1} \\ &\quad + \frac{1+\alpha}{c} \left(\frac{1}{t/\mu + t^{1/2+\delta}} \right)^{\alpha+1} \\ &\stackrel{(i)}{\geq} (1+\gamma) \left(\frac{1 - t^{-1/2+\delta} \mu - 2\mu t^{-1}}{1 + t^{-1/2+\delta} \mu} \right)^{\alpha+1} \\ &\quad + \frac{1+\alpha}{c} \left(\frac{1}{t/\mu + t^{1/2+\delta}} \right)^{\alpha+1} \\ &> 1 + \gamma \\ &> 1, \end{aligned}$$

for t large enough since as $t \rightarrow \infty$ the first term in inequality (i) goes to 1 and the second term goes to 0. From this, it immediately follows that under event \mathcal{E}_i for t large enough

$$\max_{1 \leq i \leq d_{v^*}} T_i(t) < \frac{1}{2} \left(\sum_{i=1}^{d_{v^*}} T_i(t) + 1 \right).$$

Therefore, by Lemma 1 it follows that the rumor center is unique and equals v^* . We also have that $\mathcal{E}_i \subset C_i^1$. Thus, from above and Proposition 4 we obtain

$$\begin{aligned} \lim_i \mathbf{P}(C_i^1) &\geq \lim_i \mathbf{P}(\mathcal{E}_i) \\ &= 1. \end{aligned}$$

This completes the proof of Theorem 3.4.

4.8.1. Proof of Proposition 4. We recall that Proposition 4 stated that for a rumor spreading for time t as per the SI model with a general distribution with mean spreading time μ the rumor graph on a geometric tree is full up to a distance $(t/\mu)(1-\epsilon)$ from the source and does not extend beyond $(t/\mu)(1+\epsilon)$, for $\epsilon = t^{-1/2+\delta}$ for some positive $\delta \in (0, 1/2)$. To establish this, we shall use the following well known concentration property of renewal processes. We provide its proof later for completeness.

PROPOSITION 5. Consider a renewal process $P(\cdot)$ with holding times with mean μ and finite moment generating function in the interval $(-\epsilon, \epsilon)$ for some $\epsilon > 0$. Then for any $t > 0$ and any $\gamma \in (0, \epsilon')$ for a small enough $\epsilon' > 0$, there exists a positive constant c such that

$$\mathbf{P} \left(\left| P(t) - \frac{t}{\mu} \right| \geq \frac{t\gamma}{\mu} \right) \leq 2e^{-(\gamma^2 \mu / (8c))t}$$

Now we use Proposition 5 to establish Proposition 4. Recall that the spreading time along each edge is an independent and identically distributed random variable with mean μ . Now the underlying network graph is a tree. Therefore, for any node v at distance r from source node v^* , there is a unique path (of length r) connecting v and v^* . Then, the spread of the rumor along this path can be thought of as a renewal process, say $P(t)$, and node v is infected by time t if and only if $P(t) \geq r$. Therefore, from Proposition 5 it follows that for any node v that is at distance $(t/\mu)(1-\epsilon)$ for $\epsilon = t^{-1/2+\delta}$ for some $\delta \in (0, 1/2)$ (for all t large enough),

$$\begin{aligned} \mathbf{P}(v \text{ is not rumor infected}) &\leq 2e^{-\epsilon^2 \mu t / (8c)} \\ &= 2e^{-(\mu / (8c))t^{2\delta}}. \end{aligned}$$

Now the number of such nodes at distance $(t/\mu)(1-\epsilon)$ from v^* is at most $O((t/\mu)^{\alpha+1})$ (which follows from arguments similar to those in the proof of Theorem 3.4). Therefore, by an application of the union bound it follows that

$$\begin{aligned} \mathbf{P} \left(\text{a node at distance } \frac{t}{\mu}(1-\epsilon) \text{ from } v^* \text{ isn't infected} \right) \\ &= O \left(2 \left(\frac{t}{\mu} \right)^{\alpha+1} e^{-(\mu / (8c))t^{2\delta}} \right) \\ &= O(e^{-(\mu / (8c))t^\delta}). \end{aligned}$$

Using similar argument and another application of Proposition 5, it can be argued that

$$\begin{aligned} \mathbf{P}(\text{a node at distance } t(1+\epsilon) \text{ from } v^* \text{ is infected}) \\ &= O(e^{-(\mu / (8c))t^\delta}). \end{aligned}$$

Since the rumor is a “spreading” process, if all nodes at distance r from v^* are infected, then so are all nodes at distance $r' < r$ from v^* ; if all nodes at distance r from v^* are not infected then so are all nodes at distance $r' > r$ from v^* . Therefore, it follows that with probability $1 - O(e^{-(\mu / (8c))t^\delta})$, all nodes at distance up to $(t/\mu)(1-\epsilon)$ from v^* are infected and all nodes beyond distance $(t/\mu)(1+\epsilon)$ from v^* are not infected. This completes the proof of Proposition 4.

4.8.2. Proof of Proposition 5. We wish to provide bounds on the probability of $P(t) \leq \mu t(1-\gamma)$ and $P(t) \geq \mu t(1+\gamma)$ for a renewal process $P(\cdot)$ with holding times with mean μ and finite moment generating function. Define the n th arrival time S_n as

$$S_n = \sum_{i=1}^n X_i$$

where X_i are non-negative i.i.d. random variables with a well defined moment generating function $M_X(\theta) = \mathbf{E}[\exp(\theta X)] < \infty$ for $\theta \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$ and mean $\mathbf{E}[X_i] = \mu > 0$. We can relate the arrival times to the renewal process by the following relations:

$$\mathbf{P}(P(t) \leq n) = \mathbf{P}(S_n \geq t)$$

and

$$\mathbf{P}(P(t) \geq n) = \mathbf{P}(S_n \leq t).$$

The first relation says that the probability of less than n arrivals in time t is equal to the probability that the n th arrival happens after time t . The second relation says that the probability of more than n arrivals in time t is equal to the probability that the n th arrival happens before time t .

We now bound $\mathbf{P}(S_n \geq t)$. To that end, for $\theta \in (0, \varepsilon)$ it follows from the Chernoff bound that

$$\begin{aligned} \mathbf{P}(S_n \geq t) &= \mathbf{P}(e^{\theta S_n} \geq e^{\theta t}) \\ &\leq M_X(\theta)^n e^{-\theta t}. \end{aligned}$$

We can use the following approximation for $M_X(\theta)$ which is valid for small θ , say $\theta \in (0, \varepsilon^+)$ for $0 < \varepsilon^+ \leq \varepsilon$.

$$\begin{aligned} M_X(\theta) &= 1 + \theta\mu + \theta^2 \frac{\mathbf{E}[X^2]}{2} + \theta^3 \sum_{i=3}^{\infty} \theta^{i-3} \frac{\mathbf{E}[X^i]}{i!} \\ &\leq 1 + \theta\mu + c_1\theta^2 \end{aligned}$$

for some finite positive constant c_1 . Using this along with the inequality $\log(1+x) \leq x$ for $-1 < x$, we obtain

$$\begin{aligned} \log(\mathbf{P}(S_n \geq t)) &\leq n \log(M_X(\theta)) - \theta t \\ &\leq n \log(1 + \theta\mu + c_1\theta^2) - \theta t \\ &\leq \theta(\mu n - t) + nc_1\theta^2. \end{aligned}$$

To minimize this probability, we find the θ that minimizes $\theta(\mu n - t) + nc_1\theta^2$. This happens for $\theta = (1/(2c_1))(t/n - \mu)$. We set $n = (t/\mu)(1 - \gamma)$, so the minimum value is achieved for $\theta^* = \gamma\mu/(2c_1(1 - \gamma))$. Therefore, there exists $\varepsilon_1 > 0$ so that for $\gamma \in (0, \varepsilon_1)$, the corresponding $\theta^* = \gamma\mu/(2c_1(1 - \gamma)) < \varepsilon^+$, so that the quadratic approximation of $M_X(\theta)$ is valid. Given this, we obtain

$$\begin{aligned} \log(\mathbf{P}(S_{(t/\mu)(1-\gamma)} \geq t)) &\leq -\frac{\gamma\mu}{2c_1(1-\gamma)}(\gamma t) + \frac{tc_1}{\mu}(1-\gamma) \frac{\gamma^2\mu^2}{4c_1^2(1-\gamma)^2} \\ &\leq -\frac{\gamma^2\mu t}{2c_1(1-\gamma)} + \frac{\gamma^2\mu t}{4c_1(1-\gamma)} \\ &\leq -\frac{\gamma^2\mu t}{4c_1(1-\gamma)} \\ &\leq -\frac{\gamma^2\mu t}{8c_1}. \end{aligned}$$

With this result, we obtain

$$\mathbf{P}\left(P(t) \leq \frac{t}{\mu}(1 - \gamma)\right) \leq e^{-\gamma^2\mu t/(8c_1)},$$

for any t and $\gamma \in (0, \varepsilon_1)$. For the upper bound, we have for $\theta > 0$

$$\begin{aligned} \mathbf{P}(S_n \leq t) &= \mathbf{P}(e^{-\theta S_n} \geq e^{-\theta t}) \\ &\leq M_X(-\theta)^n e^{\theta t}. \end{aligned}$$

We can use the following approximation for $M_X(-\theta)$ which is valid for small enough $\theta \in (0, \varepsilon^-)$ with $0 < \varepsilon^- \leq \varepsilon$.

$$\begin{aligned} M_X(-\theta) &= 1 - \theta\mu + \theta^2 \frac{\mathbf{E}[X^2]}{2} - \theta^3 \sum_{i=3}^{\infty} \theta^{i-3} (-1)^{i-3} \frac{\mathbf{E}[X^i]}{i!} \\ &\leq 1 - \theta\mu + c_2\theta^2 \end{aligned}$$

for some finite positive constant c_2 . Using this we obtain

$$\begin{aligned} \log(\mathbf{P}(S_n \leq t)) &\leq n \log(M_X(-\theta)) + \theta t \\ &\leq n \log(1 - \theta\mu + c_2\theta^2) + \theta t \\ &\leq \theta(t - \mu n) + nc_2\theta^2. \end{aligned}$$

To minimize this probability, we find the θ that minimizes $\theta(t - \mu n) + nc_2\theta^2$. This happens for $\theta = (1/(2c_2))(\mu - t/n)$. We set $n = (t/\mu)(1 + \gamma)$, so the minimum value is achieved for $\theta^* = \gamma\mu/(2c_2(1 + \gamma))$. There exists, $\varepsilon_2 > 0$ so that for all $\gamma \in (0, \varepsilon_2)$, $\theta^* = \gamma\mu/(2c_2(1 + \gamma)) \leq \varepsilon^-$ and thus guaranteeing the validity of quadratic approximation of $M_X(-\theta)$ that we have assumed. Subsequently, we obtain

$$\begin{aligned} \log(\mathbf{P}(S_{(t/\mu)(1+\gamma)} \leq t)) &\leq -\frac{\gamma\mu}{2c_2(1+\gamma)}(\gamma t) + \frac{tc_2}{\mu}(1+\gamma) \frac{\gamma^2\mu^2}{4c_2^2(1+\gamma)^2} \\ &\leq -\frac{\gamma^2\mu t}{2c_2(1+\gamma)} + \frac{\gamma^2\mu t}{4c_2(1+\gamma)} \\ &\leq -\frac{\gamma^2\mu t}{4c_2(1+\gamma)} \\ &\leq -\frac{\gamma^2\mu t}{8c_2}. \end{aligned}$$

With this result, we obtain

$$\mathbf{P}\left(P(t) \geq \frac{t}{\mu}(1 + \gamma)\right) \leq e^{-\gamma^2\mu t/(8c_2)},$$

for any t and $\gamma \in (0, \varepsilon_2)$.

If we set $c = \max(c_1, c_2)$ and $\varepsilon' = \min(\varepsilon_1, \varepsilon_2)$ and combine the upper and lower bounds then we obtain

$$\mathbf{P}\left(\left|P(t) - \frac{t}{\mu}\right| \geq \frac{t\gamma}{\mu}\right) \leq 2e^{-(\gamma^2\mu/(8c))t},$$

for any t and $\gamma \in (0, \varepsilon')$ with $\varepsilon' > 0$. This completes the proof of Proposition 5.

5. Conclusion

Finding the source of a rumor in a network is an important and challenging problem in many different fields. Here we characterized the performance of the rumor source estimator known as rumor centrality for generic tree graphs. Our analysis was based upon continuous time branching processes and generalized Polya's urn models. As an implication of this novel analysis method, we recovered all the previous results for regular trees from Shah and Zaman (2010) as a special case. We also showed that for rumor spreading on a random regular graphs, the probability that the estimated source is more than k hops away from the true source decays exponentially in k . Additionally, we showed that for general random trees and hence for sparse random graphs like Erdős-Rényi graphs, there is a strictly positive probability of correct rumor source detection. Thus, even though rumor centrality is an ML estimator only for a very specific setting, it is still very effective for a wide range of other graphs and spreading models. In summary, we have established the *universality* of rumor centrality as a source estimator across a variety of tree structured graphs and spreading time distributions.

Acknowledgments

Devavrat Shah would like to acknowledge conversations with David Gamarnik and Andrea Montanari at the Banff Research Institute that seeded this line of work. Both authors would like to acknowledge the support of the AFOSR Complex Networks Project, the Army Research Office MURI Award W911NF-11-1-0036 on Tomography of Social Networks and the MIT-Shell Graduate Student Fellowship.

Endnote

1. We shall call an undirected graph a *tree* if it is connected and it does not have any cycles.

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